TEACHER EDUCATION IN ALGEBRA*

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For many years we at the University of Wisconsin have been experimenting with our curriculum in algebra. Both the preparation of the students and the development of the subject itself have changed so rapidly that it has not been possible to develop a curriculum which would remain satisfactory for more than a few years. It is therefore with a feeling of little confidence that I approach this subject. I can only relate some of our experiences and experiments under the supposition that other colleges are having similar experiences, and state a few of our conclusions for whatever they may be worth.

The very spotty preparation in algebra of most of our freshmen is a major item among our troubles. Even among those students who enter the engineering college, the second year of algebra in high school is rapidly disappearing. Even more serious is the lack of competence in arithmetic and algebra of those who claim a year's preparation. Until fashions in Education swing back, there seems to be nothing effective that we can do to stem this trend. To entering engineering students we give an entrance test on elementary skills, and those students who are completely incompetent are required to spend a preliminary semester in a no-credit course where they study high school algebra. In the College of Letters and Science we give an entrance test but have no non-credit course.

Let us admit, then, that the first semester of college mathematics is largely remedial and that very little is actually learned beyond the basic skills. This course is primarily aimed at providing the tools for analytic geometry and the calculus and a minute examination of its contents does not seem to be important.

It is in the courses in analytic geometry and the calculus that the student becomes adept in algebraic manipulations. The grossly incompetent students are no longer present, and imperceptibly the problems have involved more and more complex algebraic operations so that at long last the student has acquired some skill in algebraic manipulation.

The question which we have come here to discuss is the proper content of a course or courses in algebra for a student who has just completed elementary calculus. There are many possibilities, of which I list three, because they seem to represent most common procedures:

1. A problem-solving course in permutations and combinations, probability *etc.* with advanced skills as the prime objective.

2. A development of some of the more elegant topics in classical algebra, a course in which skills and theory are kept in balance.

3. An introduction to abstract algebra.

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Before deciding which of these procedures to follow, one must examine carefully the objectives of the course. Some students at this level are aiming to take graduate work in mathematics, but more of them are not. In fact, we have graduate and undergraduate students from almost all scientific departments in the university and we cannot afford to ignore their demands for algebraic skills. It is this diversity of objectives which makes the problem difficult and calls for a compromise answer.

Since physics has become such an important science, both undergraduate and graduate students in physics have reached a level where they can be treated with mathematics majors. Most of them now enter into the spirit of a mathematics course and enjoy the theory as well as the applications. They present no particular problem.

It is frequently otherwise with advanced undergraduate and graduate students from other departments. I have some superior students from chemistry and others who are superior only in their attitude toward the Mathematics Department. They object to learning anything whose immediate application to chemistry is not evident. It is difficult to say how far our obligation to such students goes.

A few engineering students elect this course, but by and large these students keep pretty closely to their differential equations and mechanics. There is a scattering of students from agriculture, biology and the other sciences but these students are not numerous and they are inclined to be not very good. Their principal interest in algebra is as a prelude to statistics.

Approximately half of our students in junior algebra are mathematics majors. They fall sharply into two groups, graduate students who enter with a deficiency in algebra, and undergraduates working for a teacher's certificate. The latter sometimes find the competition a little rough.

To this list we might add a few students who are interested in the actuarial examinations, and a group which bids fair to increase in the next few years, namely students who are interested in computing machines.

If the University were large enough so that each of these groups could be segregated and an algebra course tailored to the individual needs of each group, the problem of the curriculum would be a small one. However, this course usually runs in at most two sections so that this cannot be done.

Probably very few mathematicians would be willing to go as far as some of our applied brothers seem to desire and omit all theory. In fact, if such an experiment were tried, I am sure that the students would become so confused that it would have to be abandoned. On the other hand, a straight course in mathematical theory is out of place with a group such as we have in our classes. A skillful teacher will know how to keep these features in balance.

With such a diversified clientele it seems undesirable to deviate too greatly from the traditional course, which I assume to be the theory of equations. There are certain definite advantages in this course. It enables one to teach skills in computing that are of definite value to students in the sciences and at the same

time develop elegant processes of reasoning that were not employed in the calculus.

While I do not believe that the student should be plunged into a course in abstract algebra at this stage in his development, I do believe that the shadow of abstract algebra falls upon this course and indicates certain changes in its organization and development which should be made.

The course which I have been giving at Wisconsin for the last couple of years is still entitled the Theory of Equations, but might more properly be called the Theory of Polynomials. This approach seems to unify the somewhat scattered topics in the theory of equations, and to give a deeper insight into the subject which is particularly valuable to those who go on in algebra and to those who contemplate teaching algebra.

You may not agree with me in bringing in about a week of the theory of numbers, but I have found it desirable, and after all the only way to check a pedagogical theory is to try it out. Only a little of the theory of numbers is required, the definitions of primes and units, scales of notation, the greatest common divisor algorithm, and the unique factorization theorem. The ideas are here presented in their simplest form free of computational difficulties. Later the corresponding concepts and theorems must be proved for polynomials. Just as a matter of experience I have found that students have trouble with this topic if it is first presented to them by way of polynomials. The introduction of this bit of number theory seems necessary in order to teach the polynomial theory regardless of its intrinsic interest.

After this bit of number theory it is easy to attack the problem of finding the integral solutions of an equation having integral coefficients, and the rational solutions of an equation having rational coefficients. Let us consider for a moment the theorem that if an equation with integral coefficients has a rational solution, when this solution is expressed in lowest terms the numerator is a divisor of the constant term of the equation. The proof depends upon the theorem in number theory that if a number divides a product and is relatively prime to one of the factors, it must divide the other factor. This students are ordinarily asked to accept as obvious, but I have seen some able students quite disturbed by it. It is proved in our bit of number theory and students having had this week of number theory really seem to understand the proof.

The plan which I have been following is to start with polynomials over the rational field, later take up polynomials over the real field, and finally polynomials over the complex field. It is well to call attention to the properties of a field, but it is not necessary to call them postulates or to introduce the notion of an abstract field. Other fields, if introduced at all, are postponed until late in the course.

The high points in the theory of polynomials are the Euclid algorithm, the unique factorization theorem, the representation of one polynomial as a polynomial in powers of a second, and the properties of the derivative in regard to multiple zeros. All of these results hold for every coefficient field but can first be

introduced for polynomials over the rational field. At this point the decomposition of a rational function into a sum of partial fractions can be rigorously established. The students are familiar with the process from integral calculus and some of them are astonished when you point out that the universality of the method had not been proved to them.

Most books on the theory of equations scrupulously avoid assuming any knowledge of the calculus on the part of the student. There may be some historical reason for this assumption, but it is no longer valid in American colleges. It is very unusual for any student to elect the theory of equations before he has had his first semester of calculus, and when it *is* done, it should not be. I am accustomed to use the derivative freely and I thus avoid the awkward situation of having to explain why the limit process is out of bounds in an algebra course.

I have always maintained that an intuitive approach should precede an abstract approach, and I feel that this is particularly true in the introduction of the real numbers. All the mystery can be taken out of the real numbers if one can show the student that the existence of an approximation process with arbitrarily small error establishes the existence of the real number. This is no place to go into the intricacies of real variable theory, but an intuitive grasp of the meaning of the real numbers is not beyond the legitimate objectives of a course in the theory of equations.

I would leave a consideration of the complex field until the last instead of introducing it at the beginning, as has lately become fashionable. To deflate the complex field from its paramount position in mathematics is one of the objectives of abstract algebra.

One of the oddities of modern convention is the introduction by many books of the complex numbers by means of Hamilton's number pairs. This is certainly in the spirit of modern abstract algebra, but it is such an isolated bit of abstraction that it seems decidedly out of place in a book that is otherwise purely intuitive. The reason why the complex numbers are frequently so introduced, and not the rational numbers nor the real numbers, is simply a matter of history. Hamilton introduced the complex numbers in this manner in 1835, but the work of Steinitz in 1910 is still probably too new for similar incorporation. In fact, I prefer to do neither, but to keep the presentation on an intuitive level at this point. The complex numbers can be nicely introduced by means of their correspondence with the points of the plane.

I would like to put in a plea here for a few days devoted to symmetric functions of the roots of an equation. Graduate students nearly always tell me that it was omitted from their course in the theory of equations on the grounds that it is of no practical value. This is a point which I am unwilling to concede. Not only is it essential for more advanced work in algebra, but it shows with great clarity how the coefficients of an equation determine the roots without showing favoritism to any one root or group of roots.

The rest of the course is pretty conventional, featuring the factorization of polynomials with real and complex coefficients. This leads to the solutions of the

cubic and quartic equations in terms of radicals. Since determinants are not a part of this course, some time is devoted to the solution of linear systems of equations by applying elementary transformations to them. This treatment of systems of equations is fast gaining in popularity. It is never more laborious than to use determinants, and in the irregular cases it is quicker and more satisfactory. A systematic treatment of systems of higher degree is also included.

It is possible to make this course more intuitive and less formal than is sometimes done. One can teach methods rather than formulas, as in the instance of finding bounds for the roots of an equation, and methods stay with a student better than a formula.

In the seventeen years that I have been at Wisconsin, this junior algebra course has vacillated between a one semester course and a two semester course several times. Now I think it has permanently become a two semester course, the second semester being devoted to matrices and determinants. Until recently it has been difficult to find a text well suited to such a course, but the publishers now seem to be erupting with many such books. This seems to indicate that other institutions than Wisconsin are planning courses along these same lines.

The recent growth of the theory of matrices in public esteem has been remarkable, but not at all surprising. Matrices are now employed in statistics, differential equations, and in computational mathematics of various kinds, to mention only a few applications. Students from other departments are coming to us with requests for work in matrices and we are having about as many students in this second semester of algebra as in the first. Probably well over half of our mathematics majors in Education take it after they have completed the Theory of Equations course.

A course in matrices at the junior level must be given with some restraint. This is not the point for an instructor or text book writer to unload all he knows. The concept of matrix must be motivated at the beginning until the student has got some idea of what it is all about.

As I have usually given the course, the first few weeks are given over to the treatment of determinants of matrices. This treatment differs very little from the old-fashioned development of determinants except in the statement of the theorems. A determinant is a number and does not have rows or columns. The matrix has the rows and columns, but a matrix is not equal to an ordinary number. Thus the accurate statement of a well-known theorem is that if two rows of a matrix are proportional, the determinant of the matrix is zero.

It seems quite natural to most students to set up the detached coefficients of a linear system of equations, and to call this array a matrix. Elementary operations on the equations, which replace the given system of equations by an equivalent system, correspond to elementary row operations on the matrix which replace it by a row-equivalent matrix. By a simple sequence of row operations every linear system may be replaced by one whose matrix is in Hermite form, whereupon the system of equations is solved. The process is applicable regardless of the number of equations or of unknowns, of whether the system is con-

sistent or inconsistent, of whether there is one solution or infinitely many.

Without doubt the next most important topic to be treated is quadratic form theory. This has familiar geometric motivations, and important applications to statistics, relativity and other fields. There are some computational difficulties, so that the first problems to be worked should be chosen so as to minimize these difficulties. The problem should be formulated in matric notation as well as in the notation of quadratic forms, and pretty soon the student will be dealing with matrices as abstract entities.

This is a most excellent place to point out the significance of the concept of field, and the fundamental differences between the rational, real and complex fields. The concept of invariant is well illustrated by the rank and signature of a quadratic form.

Custom now calls for a treatment of orthogonal matrices, and the orthogonal reduction of a real symmetric matrix to diagonal form. For third order matrices this is the principal-axis transformation of solid analytic geometry. To students who have clawed rather ineffectually at this problem in a geometry class, the complete solution of the problem is revealing.

Such then, is the content of our course 115 which most candidates for the teacher's certificate with major in mathematics take at Wisconsin. It is a fairly substantial course, but I believe I speak for my colleagues when I say that we are never entirely satisfied with it. There are so many things that it does not contain that it would be good for a teacher of mathematics to know. It is true that a few of the concepts of abstract algebra are worked into it, such as field, ring and group, but so much is omitted that a modern student of mathematics should know.

You are probably about to ask the question, why do we not put the prospective teachers into a separate section and give them abstract algebra. We have tried it without much success. Most of our Education students are not in the genius class and they absorb the abstract point of view very slowly. Also they do not have the basic algebraic facility nor the fund of illustrative examples to make the subject meaningful. It looks as if we shall have to be content to keep our junior algebra course at a somewhat old-fashioned level and surreptitiously to work in as many abstract concepts as the material allows.

But to my mind the situation indicates pretty strongly that a well qualified teacher of mathematics in high school should have a master's degree or at least a little summer work beyond the bachelor's degree. This is not a revolutionary suggestion, but the really radical and communistic suggestion that I am going to make is that the master's work should include at least one course in the subject which the candidate teaches. Every summer we have thousands of teachers in our Summer School, a dozen or so of whom ever go inside North Hall.

We have for about eight years offered courses in the Summer Session expressly for high school teachers of mathematics. They are entitled Foundations of Arithmetic and Foundations of Algebra, and are given in alternate years. We had believed that as time went on these courses would attract an increasing

clientele, but the classes remain small while hundreds of teachers go next door for courses in the Principles of Education.

The course in Foundations of Arithmetic exposes the student to some modern mathematics. The natural numbers are introduced by means of the Peano postulates, the rational integers as pairs of natural numbers, the rational numbers as pairs of rational integers, the real numbers as sequences of rational numbers, the complex numbers as pairs of real numbers, and finally the real quaternions as tetrads of real numbers. This is Hamilton's idea carried to a logical conclusion. Within this framework there is opportunity to take up scales of notation other than the decimal scale, proof of the rule of casting out nines, and a critical examination of all of the elementary operations of arithmetic including cube root. The student is treated to proofs of all the fundamental laws of arithmetic including the fact that zero times one is zero. It is true that the proof is not one that the teacher can show to an elementary student, but at least the teacher knows that it is true.

The course has been very successful with those whom we have been able to persuade to take it. The point of view has been rather lofty but we have not required a very high level of attainment on the part of the student teacher. We have felt that it was sufficient to get across the general idea without requiring the memorization of too many details. In fact, there was frequently marked enthusiasm by the class when we finally proved some point which had previously bothered these teachers when they had attempted to present the subject to their own elementary classes.

The other course, given in alternate summers with the Foundations of Arithmetic, was aimed at strengthening the foundations of elementary algebra. In my own opinion it was not as good a course as the first. Almost without exception the students registering for the course had not had the theory of equations, so that we were up against the situation described earlier · · · no technical competence. As a result this course has overlapped considerably with the theory of equations.

I am afraid that in this course I have spent some time presenting antidotes to some of the ideas which the students have picked up in methods courses—not at the University of Wisconsin, I hasten to add. I shall take just one example. In a methods course, there are various distinct methods for solving a quadratic equation: factoring, completing the square, and using the formula. But from the point of view of a person who is interested in fundamental principles, there is just one: a product is zero if and only if one of its factors is zero.

I wonder if it is really a good idea to break down a subject such as algebra into a large number of steps, methods and processes. Perhaps it is the only way with very unintelligent students. But with students of medium intelligence or above, I think it is a step in the wrong direction. The number of fundamental principles in algebra is small and if these principles can be kept to the forefront, and the student made to realize that he can rely on common sense, the whole subject should appear vastly simplified. I have often said that if I were teaching

a class in high school mathematics I would want a room with large blank wall spaces on either side. In one of these spaces I would have printed in large letters, "A product is equal to zero if and only if at least one of the factors is zero." In the other blank space I would print, "A fraction is equal to zero if and only if its numerator is zero and its denominator is not zero." A large amount of algebraic knowledge is locked up in these two statements.

One of the commonest misconceptions that I have found among these teachers relates to the concept of axiom or postulate. I suppose this is quite natural, for the concept is tricky and many books are quite wrong or, let us say, out of date in their usage. Actually there are very few genuine postulates in elementary non-geometric mathematics. The Postulates of Peano or some equivalent assumptions seem to be necessary for the introduction of the positive integers, but after that it is simply a matter of defining a new type of number in terms of those already defined, defining operations upon them, and then of finding out what properties these new numbers and operations possess. To call the commutative law or the distributive law an axiom is dishonest. Either it is true or not true for the numbers with which you work, and you have no choice whether you will assume it or not. The fact that you are not able to present the proof to an elementary class does not justify you in calling it an axiom. It is better to merely say that the numbers with which you are working have this property. If some bright student challenges your statement, tell him honestly that the proof is too tricky for presentation to an elementary class.

May I put in a little political plug before closing? In these days when the master's degree is coming more and more to be required for advancement in the profession, is it too much to ask that half of the work toward the degree be in the subject which the candidate expects to teach or in some related subject? How infinitely more capable and more inspiring would a teacher of algebra be if he were not only conversant with classical algebra as exemplified by a year's work in algebra beyond the calculus, but if he had also taken a graduate course in abstract algebra. I do not wish to imply, of course, that a teacher should start presenting abstract algebra to his pupils in high school, but I am sure that he would radiate confidence and enthusiasm to a degree that would make him a trusted and admired teacher.

Frankly I do not know how such a program could be put over. I assume that those high school teachers of mathematics who take work in mathematics in summer sessions are among the best and most enthusiastic in the high school field. But most of them are not very good scholars. They are not as good as our run-of-mine juniors. Probably they once were, but after being out of the atmosphere for a few years they have deteriorated. What the others are like I hate to think.

But I do not see how we are going to persuade high school teachers of mathematics to take more mathematics in the summer time. School boards frequently encourage teachers to attend summer school, but the subjects that they shall take is seldom specified or, if it is specified, it is in the Principles of Education.

And a more delightful summer vacation may be had by by-passing subject matter courses.

Possibly some good might be done if the books on How to Teach Mathematics were of a little higher calibre. An Educational expert who does not know any mathematics beyond elementary algebra is not in a position to write such a book. It is perhaps also true that a research mathematician is not able to restrain himself sufficiently to keep the book within the comprehension of his audience, in which case his book will simply not be read. What we really need is another J. W. Young and an American David Hilbert who will write elegant but elementary expositions for the benefit of those who are not very experienced in the field, men who are not afraid to tell less than they know about a subject in fear that some book reviewer will misjudge the depth of their scholarship.

ON FACTORIZATION OF POLYNOMIALS

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1. Introduction. The problem of determining, in a finite number of steps, the factorization over the rational field of a given numerical polynomial has been solved by Kronecker. His method is based on the fact that, when the polynomials $f(x)$ and $g(x)$ have integral coefficients, $f(x)$ can be divisible by $g(x)$ only if $f(n)$ is divisible by $g(n)$ for any integer n. Even when one takes advantage of the improvements in Kronecker's method which have been developed by various writers, its application usually involves lengthy and cumbersome calculations. In this paper we present an alternative procedure which focuses attention on the coefficients of the polynomial, rather than on its values at the integers. We believe that our method requires less effort than Kronecker's.

2. The ∇-functions.
Let

$$f(x) = x^n + \sum_{j=1}^{n} (-1)^j s_j x^{n-j}$$

be a polynomial with rational integral coefficients and leading coefficient one. s_j is the elementary symmetric function of order j of the roots $r_1, r_2, \cdots, r_n$ of $f(x)$. We define the integer $\nabla_{j,k} f(x)$ by means of the equation,

$$\nabla_{j,k} f(x) = \prod \sigma_j(r_1', r_2', \cdots, r_k'),$$

where $r_1', r_2', \cdots, r_k'$ is a particular combination of k of the n roots of $f(x)$, $\sigma_j(r_1', r_2', \cdots, r_k')$ is the elementary symmetric function of order j of these k roots, and the product is taken over all combinations of the roots of $f(x)$ taken k at a time. The symmetric function theorem tells us that $\nabla_{j,k}f(x)$ may be written as a polynomial in $s_1, s_2, \cdots, s_n$ with rational integral coefficients. The calculation of this polynomial is rather tedious, but it is something that needs to be done only once for particular values of j, k and n. We give here a table of $\nabla_{j,k}$ for $n=4$ and $n=5$.

$$n = 4 \qquad f(x) = x^4 - s_1x^3 + s_2x^2 - s_3x + s_4$$

$$\nabla_{1,1}f(x) = s_4$$

$$\nabla_{1,2}f(x) = s_1s_2s_3 - s_3^2 - s_1^2s_4$$

$$\nabla_{1,3}f(x) = s_1^2s_2 - s_1s_3 + s_4$$

$$\nabla_{2,3}f(x) = s_2s_3^2 - s_1s_3s_4 + s_4^2$$

$$n = 5 \qquad f(x) = x^5 - s_1x^4 + s_2x^3 - s_3x^2 + s_4x - s_5$$

$$\nabla_{1,1}f(x) = s_5$$

$$\nabla_{1,2}f(x) = s_1s_2s_3s_4 - s_3^2s_4 - s_1^2s_4^2 - s_5^2 + 2s_1s_4s_5 + s_2s_3s_5 - s_1^2s_2s_5$$

$$\nabla_{1,3}f(x) = (s_1^2s_2 - s_1s_3 + s_4)(s_1s_2s_3 - s_3^2 - s_1^2s_4) - s_5^2 - 2s_1s_4s_5^2 + s_2s_3s_5^2 + 3s_1s_3^2s_5^2 - s_1^3s_2s_5^2 + s_1^5s_5^2$$

$$\nabla_{1,4}f(x) = s_1^3s_2 - s_1^2s_3 + s_1s_4 - s_5$$

$$\nabla_{2,3}f(x) = (s_3^2s_4 - s_2s_4^2s_5 + s_1s_5^2)(s_2s_3s_4 - s_2^2s_5 - s_1^2s_4^2) - s_5^4 - 2s_1s_4^3s_5^3 + s_2s_3^3s_5^3 + 3s_2^2s_4^2s_5^2 - s_3^3s_4^2s_5^2 + s_4^5s_5^2$$

$$\nabla_{2,4}f(x) = s_2^2s_3^2 - s_1s_2s_3s_4^2 + s_2^2s_4^2 + s_5^2 - s_1s_4s_5^2 - 2s_2s_3s_5^2 + s_1^2s_3s_5^2$$

$$\nabla_{3,4}f(x) = s_3^3s_4^2 - s_2s_4^3s_5^2 + s_1s_4^2s_5^2 - s_5^3$$

Let $g(x)=x^k+\sum_{j=1}^{k}(-1)^jt_jx^{k-j}$ be a divisor of $f(x)$ with rational integral coefficients. Then t_j is a divisor of $\nabla_{j,k}f(x)$. For the quotient $\nabla_{j,k}f(x)/t_j$ is on the one hand rational, since both numerator and denominator are, and on the other hand, an algebraic integer, since it is the product of algebraic integers.

3. The method. We wish to determine whether or not $f(x)$ has any rational factors of degree k. There is no loss of generality in supposing that $f(z)$ has integral coefficients with leading coefficient unity, for a simple substitution will always put $f(x)$ in this form. Suppose that

$$f(x) = g(x)h(x)$$

where

$$h(x) = x^{n-k} + \sum_{j=1}^{n-k}(-1)^jt_j'x^{n-k-j}$$

and $f(x)$ and $g(x)$ are as before. Then t_1 is a divisor of $\nabla_{1,k}f(x)$, t_1' is a divisor of $\nabla_{1,n-k}f(x)$ and

$$t_1 + t_1' = s_1. \tag{1}$$

If no pairs of divisors of $\nabla_{1,k}$ and $\nabla_{1,n-k}$ satisfy (1), then $f(x)$ has no factor of degree k. If several such pairs (τ_1, τ_1') exist, it is possible to eliminate some of them by the following process. Compute $f(x+a)$ for some suitable integer a. To the pair of divisors (τ_1, τ_1') of $\nabla_{1,k}f(x)$ and $\nabla_{1,n-k}f(x)$ must correspond the pair of divisors $(\tau_1+ka, \tau_1'+(n-k)a)$ of $\nabla_{1,k}f(x+a)$ and $\nabla_{1,n-k}f(x+a)$. This is readily seen by replacing x by $x+a$ in $g(x)$ and $h(x)$. If either τ_1+ka does not divide $\nabla_{1,k}f(x+a)$ or $\tau_1'+(n-k)a$ does not divide $\nabla_{1,n-k}f(x+a)$, then the original pair (τ_1, τ_1') may be removed from consideration. It may happen that $f(x)$ has a factor of degree k in some extension of the rational field and that the first coefficient, τ_1, of this factor is rational. In this case, no choice of a would eliminate the corresponding pair (τ_1, τ_1'). As an example, consider the polynomial

$$x^4 + 2x^3 + x^2 + 1 = (x^2 + x + i)(x^2 + x - i).$$

This polynomial is actually irreducible over the rational field, as would become apparent after the application of the next step of our method.

If $M = \max |s_i|$, then it is easy to show that $|\tau_i| \leqq M+1$, $i=1, 2, \cdots, n$. Hence $|t_1| \leqq k(M+1)$ and $|t_1'| \leqq (n-k)(M+1)$. It is possible to eliminate some of the pairs (τ_1, τ_1') by means of these inequalities. Similar inequalities may be derived for the subsequent coefficients.

If, after a few choices of a, one or more pairs (τ_1, τ_1') are not eliminated, it is probable that $f(x)$ has several factors of degree k. Take a particular pair (τ_1, τ_1') and call it (t_1, t_1'). (In cases where $f(x)$ has just one factor of degree k, (t_1, t_1') will usually be determined uniquely.) We then proceed to the determination of t_2 and t_2'. Now t_2 is a divisor of $\nabla_{2,k}f(x)$, t_2' is a divisor of $\nabla_{2,n-k}f(x)$ and

$$t_2 + t_2' = s_2 - t_1t_1'. \tag{2}$$

We search for pairs of divisors of $\nabla_{2,k}f(x)$ and $\nabla_{2,n-k}f(x)$ which satisfy (2). Again, if a number of such pairs (τ_2, τ_2') exist, many can be eliminated by considering $f(x+a)$. Then, as one sees by replacing x by $x+a$ in $f(x)$ and $g(x)$, $\tau_2 - t_1(k-1)a + \frac{1}{2}k(k-1)a^2$ must divide $\nabla_{2,k}f(x+a)$ and $\tau_2' - t_1'(n-k-1)a + \frac{1}{2}(n-k)(n-k-1)a^2$ must divide $\nabla_{2,n-k}f(x+a)$.

After t_2 and t_2' have been found, one continues in an exactly similar fashion and determines the remaining coefficients. Our method may then be applied to the factors $g(x)$ and $h(x)$ until finally $f(x)$ is broken down into its irreducible factors. Ordinarily it is not necessary to use our method for the calculation of all the coefficients of $g(x)$ and $h(x)$. Frequently simpler devices suggest themselves.

4. Examples. We illustrate the foregoing with two examples.

Example 1. $f(x)=x^4-2x^3+6x^2+5x+6$.

Our method could be used to test for linear factors, ($k=1$), but it does not involve any great saving of labor; in fact, in most cases the traditional procedure, to which ours is similar, is better. It is easily seen that in this instance $f(x)$ has no linear factors.

We now look for quadratic factors. Using the table, we see that $\nabla_{1,2}f(x) = -109$. The only possible values of t_1 and t_1', are -1 and -1. But $f(x-1) = x^4-6x^3+18x^2-17x+10$ and $\nabla_{1,2}f(x-1)=1187$, which is not divisible by -3. Hence $f(x)$ has no rational quadratic factors and is consequently irreducible. These latter calculations might have been somewhat simplified by computing $f(x-1)$ and $\nabla_{1,2}f(x-1)$ modulo 3.

A better way of eliminating the pair $(-1, -1)$ is the following. If $f(x)$ has a pair of quadratic factors, (x^2-x+t_2) and (x^2-x+t_2'), then examination of the last two coefficients shows that we must have $(-x+t_2)(-x+t_2') = x^2+5x+6$, whence $t_2=-3$ and $t_2'=-2$. But $(x^2-x-3)(x^2-x-2) \neq f(x)$, since the coefficients of x^2 do not agree.

Example 2. $f(x)=x^5-3x^4-x^3+x^2+5x+15$.

As before, there are no linear factors. We seek quadratic factors. From the table we have

$$\nabla_{1,2}f(x) = -\ 860 = -\ 2^2\cdot 5\cdot 43,$$

$$\nabla_{1,3}f(x) = -\ 3392 = -\ 2^6\cdot 53.$$

The only pairs of divisors of $\nabla_{1,2}f(x)$ and $\nabla_{1,3}f(x)$ which satisfy the condition $\tau_1+\tau_1'=-3$ are $(1, -4)$, $(5, -8)$, $(-1, -2)$, $(-2, -1)$, $(-4, 1)$ and $(-5, 2)$. Now

$$f(x+1) = x^5 + 2x^4 - 3x^3 - 10x^2 - 3x + 18$$

and

$$\nabla_{1,2}f(x+1) = -\ 240 = -\ 2^4\cdot 3\cdot 5,$$

$$\nabla_{1,3}f(x+1) = -\ 860 = -\ 2^2\cdot 5\cdot 43.$$

This eliminates the pair $(5, -8)$, since $\nabla_{1,2}f(x+1)$ is not divisible by $5+2=7$, and the pair $(-2, -1)$, since $\nabla_{1,3}f(x+1)\neq 0$. We try

$$f(x-1) = x^5 - 8x^4 + 21x^3 - 24x^2 + 17x + 8.$$

Here

$$\nabla_{1,2}f(x-1) = 2^3\cdot 3^5$$

while

$$\nabla_{1,3}f(x-1) = 2^6\cdot 5\cdot 79\cdot 101.$$

This eliminates $(-5, 2)$ since $\nabla_{1,3}f(x-1)$ is not divisible by $-5+2(-1)=-7$ and $(1, -4)$ since $\nabla_{1,3}f(x-1)$ is not divisible by $-4+3(-1)=-7$. There remain only $(1, -2)$ and $(4, -1)$; the latter may be eliminated by considering $f(x+2)$ and observing that $\nabla_{1,2}f(x+2)\neq 0$.

We investigate t_2 and t_2'. Notice that, from (2), $t_2+t_2'=-3$. We find that $\nabla_{2,2}f(x)=0$. Simple field-theoretic considerations show that, if $f(x)$ is reducible, we must have $t_2'=0$. Then we are forced to put $t_2=-3$, $t_3'=-5$ and $f(x)=(x^2-x-3)(x^3-2x^2-5)$. It is readily seen that this is a factorization of $f(x)$. Since neither factor has any rational roots, $f(x)$ can be factored no further.

5. Related methods. A method closely related to ours was introduced by Sebastião e Silva.* He observes that if $f(x)$ has a factor, $g(x)$, of degree k, then the equations

$$\prod (t-\sigma_j(r_1', r_2', \cdots, r_k')) = 0, \qquad j = 1, 2, \cdots, k,$$

where the products are taken over all combinations of the roots of $f(x)$ taken k at a time, must have rational roots. These roots, if they exist, may be found by the traditional method. One notices that our $\nabla_{j,k}f(x)$ are simply the constant terms of these equations. Our method does not require the computation of the remaining coefficients. On the other hand, Sebastião e Silva's method does not require the replacement of x by $x+a$ and a repetition of the process, as ours frequently does.

Ore† has remarked that if $f(x)$ has a quadratic factor $g(x)=x^2-t_1x+t_2$, then the discriminant of $f(x)$ must be divisible by $t_1^2-4t_2$, the discriminant of $g(x)$. He uses this fact to restrict the possible values of t_1. This method does not seem as direct as ours, and ordinarily will require the calculation and factorization of larger numbers. However, in many investigations, one has to determine the factors of the discriminant of $f(x)$ for other purposes.

* Jose Sebastião e Silva, Problems concerning rational functions of the roots of an algebraic equation. Portugaliae Math. 2, 1941, pp. 20–35.

† Oystein Ore, A note on factorization of polynomials—Revista Ci. Lima 41, 1939, pp. 587–592.

MATHEMATICS IN SCHOOL AND COLLEGE*

Editorial Note: The following article is an extract from the book *General Education in School and College*, Harvard University Press, 1952, pp. 52–57. This book is the outcome of a study designed to improve the integration between school and college undertaken by members of the faculties of Andover, Exeter, Lawrenceville, Harvard, Princeton, and Yale. The portion of the report dealing with mathematics was based, in part, on a survey conducted by Richard S. Pieters (Andover) and on a panel discussion in which the committee was joined by A. W. Tucker (Princeton), E. G. Begle (Yale), Peter J. Kiernan (Lawrenceville), Ransom Lynch (Exeter), Winfield M. Sides and Richard S. Pieters (Andover).

A reading of the entire report reveals that the recommendations below were framed with special reference to the better-than-average student. Also it should be noted that the conclusions are intended to apply to the relationships between the six institutions represented and are not necessarily valid in other situations. Your editor believes, however, that the basic ideas expressed can be of great value to mathematicians in other schools and universities who may be interested in incorporating suitable modifications of these suggestions into their own curricula.

No subject is more properly a major part of secondary education than mathematics. None has a more distinguished history or a finer tradition of teaching. Perhaps the very excellence of the topic has helped, in recent decades, to keep the content and order of its teaching largely unexamined. One of the most remarkable of our sessions was the one in which we consulted with a group of first-rate school and college teachers of mathematics and discovered, as the evening progressed, a very high degree of consensus on the view that school offerings in mathematics are ready for drastic alteration and improvement.

At present the basic four-year course of mathematics in the schools we have studied covers two years of algebra, one of plane geometry, and one of solid geometry and trigonometry (the latter year elective). A small group of boys (less than one in five) are advanced more rapidly in two of the schools so that in their last year they are introduced to the calculus in a course roughly comparable to first-year college calculus. This basic four-year curriculum has the sanction of the ages, and there can be no doubt that every subject in it has a value of its own. But the Committee was still more impressed by the value of what is usually omitted, and by the fact that in some cases the words of principle seem to be obscured by the trees of constant repetition and problem-solving.

Mathematics is the rigorous application of notions that are vaguely apparent even to the nonmathematical; its wonder lies in its demonstration of the extraordinary power which comes from thinking closely and connectedly. It is not remarkable that an apple drops with growing speed from tree to ground, but what Newton did to this idea is one of the heroic advances of the human mind. The means by which he did it are not easily understood; the power of abstract thinking is matched by its difficulty. This is true for all the basic ideas of mathematics, yet it is just these basic ideas that are of the greatest value to the student. As the Harvard Report has noted, "It is unfortunately true that those

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aspects of algebra and geometry that are of greatest interest in general education are also more difficult to teach, and are much harder for the student to grasp, than are the technical skills of mathematical manipulation."* From every point of view—that of the college mathematics teacher as well as that of general education—we are persuaded that the greatest business of the school's mathematics curriculum should be to communicate as many of these central concepts as it can; we think the better student can do much more of this than he has been doing, and we are certain that he should try.

But from the point of view of a crowded curriculum we are convinced that each branch of mathematics operates under a law of rapidly diminishing returns. Once the basic notions are solidly understood—and this will require drill as well as thought—we think it is unwise to linger in loving elaboration of a set of ideas grown familiar. Of course it is possible to design problems of bewildering complexity in every subject from long division to trigonometry; it is also a waste of time. We do not press the paradox, but we suggest that it is almost true that the better the student, the fewer problems he should be asked to solve.

The impact of these notions upon the present mathematical curriculum is heavy, for large parts of it turn out to be relatively unhelpful elaborations of principles which are better taught in other ways. The greatest single offender in this sense is solid geometry. It is a beautiful subject, but in the strictly mathematical sense it is an elaboration of plane geometry, and elaboration is not the point of mathematics. The real value of solid geometry lies outside its mathematics, in the fact that it tends to develop a general sense of spatial reality. This can be done more briefly and more effectively, we think, if the effort to develop a systematic structure is abandoned. If generally adopted, this single revision would save nearly half a year in the standard school curriculum.

The example of solid geometry can be repeated, on a smaller scale, in the teaching of algebra, plane geometry, and trigonometry. Among the topics which seem appropriate for condensation or omission are complex numbers,† determinants, logarithmic solutions of triangles, and the geometry of the circle. These are suggestions only; much planning and experience will be needed before teachers of mathematics can determine where to modify and revise their present practice, and it is by no means certain that the problem has a unique solution. All that we assert, and we do so with the authority of our professional advisers, is that much can be squeezed out, to the positive advantage of the basic notions which are now taught.

If we are correct in claiming that there is excess fat on the body mathematical, it is good, in itself, to trim the present curriculum; for it is one of the basic conclusions of our whole study that it is bad to do in two terms what should be

* General Education in a Free Society (Harvard University Press, 1945), p. 163.

† Professor Tucker has suggested to the editor that this reference to complex numbers may have slipped in by accident or in any event was not sufficiently considered by the committee which prepared the report.

done in one. But we are not urging a cut in the traditional mathematics course because we want less time for mathematics. What we are trying to do is at once to bring into visible relief the great ideas of traditional algebra and geometry, and at the same time to make room for two additional sets of notions—those related to the calculus and to statistics. This may be too much to ask, but before we consider the difficulties let us briefly urge the importance of each of these two topics.

The importance of the calculus for all scientists and engineers needs no argument. Its respectability as a discipline in liberal education is equally plain; it is the standard freshman course in the best of our colleges. The only thing new in our position is the suggestion that the elements of this subject can be presented to a large percentage of well-trained schoolboys in the 12th grade. Not very many, perhaps, can meet the standards of a college course, but that is not our first object. What we are eager to do is to get into the minds of the student who is *not* headed for a career in science or mathematics some sense of the power and meaning of the calculus. We are firmly persuaded that this can be done in school, and it seems to us to have a value, in general education, that is greater by several orders of magnitude than the value of drill in the elaboration of solid geometry or determinants. The student who has once grasped the meaning of differentiation and integration sees the world afterward in a larger and more significant way; his exposure to science takes a different shape; his sense of space and motion is enlarged. Obviously not all students take this meaning from a single course in the calculus; not all teachers try to teach it. But for those who can, the opportunity is too great to be sacrificed to anything but necessity, and we are bound to conclude that tradition, not necessity, is what currently limits the teaching of elementary calculus in school.

There is a further and most practical advantage in an expanded teaching of the calculus in school. It is that a student of physics, however good his training in school, finds it extremely difficult to handle advanced physics courses unless he has been introduced to the calculus. The prospective scientist can clear his way in both physics and mathematics if he does solid work in the calculus in school.

In principle, the nonscientist with school training in the calculus might reap a parallel advantage; in his study of science in college he could go much further into the meaning of physics than would otherwise be possible. It is a serious misunderstanding to suppose that general education and mathematical physics are opposites. The best of the college courses for the nonscientist make considerable demands on the mathematical equipment of their students, and their teachers would be happy to build on a knowledge of elementary calculus. But such a development may take time.

To some of our consultants, and in some respects to our Committee as well, the case for statistics is even more powerful than the case for the calculus. The notions of probability, correlation, and sampling are among the fundamentals

of modern social measurement. And since we live in the age of polls, an awareness of the real meaning of these notions is a protection to the consumer as well as a necessity for the producer of information. Moreover, there is in all statistics a salutary concern for the uncertain and the incomplete—for the gray that is real more than for the black and white that is abstraction. It is well for the student to learn both that mathematics has uncertainty and that uncertainty can be mathematically treated. This knowledge is important in many fields; teachers of science and teachers of history alike have their troubles with students who are persuaded that all reasoning is geometrical and all evidence conclusive. All in all, if we had a curriculum to build from the ground up, we cannot suppose that it would omit statistics from a general education.

If we could find the time, we would urge that the standard school curriculum include about one year of the calculus (with a minimal framework of analytic geometry) and half a year of statistics. And if all else were equal, we would urge that the statistics follow the calculus; it has much more meaning and power that way. But time is not easy to find, and all other things are not equal. The teaching of statistics in school can hardly grow very fast until there is a body of teachers with experience in the subject and some core of knowledge on which to build the right kind of course. Moreover, there is a real advantage in leaving the whole of the 12th grade for the calculus—this is the topic that leads directly to advanced work in college, and it is a well-tested and coherent unit which is best taught in a single academic year. Finally, nearly all good school departments of mathematics are well-equipped to teach the calculus; they have been teaching it, on a limited scale, for many years. On balance, therefore, we recommend that the schools should move toward a curriculum in which the basic 12th grade course is the introduction to the calculus. At the same time we hope that there will be intensive experimentation with the teaching of some of the basic concepts of statistics, and we think there is room for this in the second year's study of algebra. Nor do we exclude the possibility that some schools may wish to offer statistics as an alternative to the calculus, or even as an additional elective.

It is our conclusion that the school mathematics curriculum can and should be redesigned to include new areas of instruction; and we think that when this has been done, college mathematics should be a subject for scientists, mathematicians, and really talented amateurs of the topic. For others, there is plenty in the basic course we have outlined, and their college work should be in other fields. And this we feel is as it should be; fundamental mathematics of the sort we have been discussing is taught better—and learned better—in the schools than in the colleges.

SOME REFINEMENTS IN THE THEORY OF SPECIALIZED SPACE CURVES

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1. Introduction. The general space curve of this discussion has the parametric vector representation

$$\mathbf{x} = \mathbf{x}(t). \tag{1}$$

Equation (1) consists of the triple of scalar functions

$$x_i = x_i(t); \qquad i = 1, 2, 3, \tag{1'}$$

where the x_i are projections on the axes of a rectangular coordinate system in Cartesian three-space. We require that the domain of equation (1) be a real interval $a \leqq t \leqq b$ and that the functions (1′) be real- and single-valued on this domain. Also, from time to time, we specify that functions (1′) belong to class C^n; $n = 1, 2, 3, \cdots$, that is, to the class of functions with continuous derivatives of order n. Finally, for any two vectors $\mathbf{a}$ and $\mathbf{b}$, the inner or scalar product $(\mathbf{a} \cdot \mathbf{b})$ and the outer or vector product $\mathbf{a} \times \mathbf{b}$ in this discussion have their usual definitions and properties.

Standard texts of differential geometry usually eliminate from the class of space curves all equations of the form (1) which possess singular points. That is, the authors specify that not all the $x_i'(t)$ vanish at any point, when they discuss straight lines and similarly that $\mathbf{x}'(t) \times \mathbf{x}''(t)$ never vanishes when they consider plane curves. Under these restrictions, the standard texts are able to prove that the necessary and sufficient conditions for an equation to represent a straight line are equivalent. The texts give a similar proof for plane curves [1, 4, 7].

The writer suggests the need for a discussion of rectilinearity and planeness which admits to consideration any equation of the form (1) with the properties ascribed to it in the first paragraph. Let us compare necessary and sufficient conditions for the two geometrical properties within this broad class of equations.

2. Straight lines. Many of the rectifiable curves of differential geometry have a "natural" parametric representation in the sense that the parameter s of this representation expresses the arc length of these curves. Proof of the existence of this representation for a given curve $\mathbf{x}(t)$ requires that the functions (1′) be of class C^1, in which case the curve is rectifiable, and that always at least one of the first derivatives of these functions be different from zero [7, pp. 72–73 and 44].

This arc length parametrization $\mathbf{x}(s)$ is mentioned because, within the class of curves which are expressed in this form, the necessary and sufficient conditions for rectilinearity are equivalent. Thus for these special curves only, we follow the pattern of the standard texts in proving

THEOREM I. *The curve* $\mathbf{x}(s)$ *of class* C^2 *and expressed in terms of its arc length*

is a straight line if and only if

$$\mathbf{x}''(s) \equiv 0.$$

If $\mathbf{x}(s)$ is a straight line, it will have the arc length representation of class C^2

$$\mathbf{x} = \mathbf{r} + \mathbf{a}s$$

where $\mathbf{r}$ and $\mathbf{a}$ are constant vectors. Here, arc length s is measured along the line from $\mathbf{r}$ in direction $\mathbf{a}$. We have

$$\mathbf{x}'(s) = \mathbf{a} \not\equiv 0$$
$$\mathbf{x}''(s) \equiv 0.$$

Conversely, if $\mathbf{x}(s)$ is known to be of class C^2 and to satisfy this last relationship, successive component-wise integrations yield

$$\mathbf{x} = \mathbf{r} + \mathbf{a}s.$$

This is interpreted geometrically as a straight line with s as arc length, and the theorem is proved.

Our departure from the standard discussions comes when the curve is expressed as a straight line by using the arbitrary parameter t to write

$$\mathbf{x} = \mathbf{x}(t) = \mathbf{r} + \mathbf{a}\phi(t)$$

in terms of the point $\mathbf{r}$, direction $\mathbf{a}$, and scalar function $\phi(t)$. We assume that $\mathbf{x}(t)$ is of class C^2 so that, differentiating,

$$\mathbf{x}'(t) = \mathbf{a}\phi'(t)$$
$$\mathbf{x}''(t) = \mathbf{a}\phi''(t).$$

Since we exclude points from the class of straight lines,

$$\mathbf{x}'(t) \not\equiv 0.$$

Otherwise $\mathbf{x}(t)$ would reduce to a constant vector, or point. From the expressions for $\mathbf{x}'$ and $\mathbf{x}''$,

$$\mathbf{x}'(t) \times \mathbf{x}''(t) = \mathbf{a} \times \mathbf{a}[\phi'(t)\phi''(t)] \equiv 0,$$

and we state this result in

THEOREM II. *If*:

1. $\mathbf{x}(t)$ is of class C^2,
2. $\mathbf{x}(t)$ *represents a straight line,*

then:

A. $\mathbf{x}'(t) \not\equiv 0$,

B. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$.

The example

$$\mathbf{x} = (-t^3, 0, 0), \quad t \leqq 0; \qquad \mathbf{x} = (0, t^3, 0), \quad t > 0$$

demonstrates that the converse of this last theorem is not true. Conclusions A and B above hold for the example which is of class C^2. Yet rectilinearity fails at the origin, where $\mathbf{x}'(t)$ vanishes. Evidently some modification is needed in stating sufficient conditions for rectilinearity for the class of space curves under consideration. As a first approach to the problem, we shall assume differentiability of class C^2, conclusion B of Theorem II, and the additional hypothesis

(2) $\mathbf{x}'(t)$ never vanishes.

Under these conditions, $\mathbf{x}'$ is parallel to $\mathbf{x}''$ or $\mathbf{x}'' = 0$ so that, for some scalar function $k(t)$,

$$\mathbf{x}''(t) = k(t)\mathbf{x}'(t),$$

which relation may be expressed as

$$\frac{d}{dt}\, x_i'(t) = k(t)x_i'(t); \qquad i = 1, 2, 3.$$

Condition (2) implies $k(t)$ is continuous so that, from existence and uniqueness theorems of differential equations [5], $x_i'(t)$ is given by

$$x_i'(t) = a_i e^{\int k(t)dt}; \qquad a_i = \frac{x_i'(t_0)}{\left[e^{\int k(t)dt}\right]_{t=t_0}},$$

where t_0 is an arbitrary value of t. Similarly,

$$x_i(t) = a_i f(t) + r_i; \qquad f(t) = \int e^{\int k(t)dt} dt.$$

This has the form of the equation for a straight line, from which we deduced Theorem II. $\mathbf{x}(t)$ cannot reduce to a point by condition (2). We have

THEOREM III. *If*:
1. $\mathbf{x}(t)$ *is of class* C^2,
2. $\mathbf{x}'(t)$ *never vanishes*,
3. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$,

then equation (1) *represents a straight line.*

This theorem expresses the fact that sufficient conditions for rectilinearity must be stronger than the necessary conditions of Theorem II. The reader may further satisfy himself on this point by proving that any broken line can be represented by an equation of the form (1) with $\mathbf{x}(t)$ of class C^2 and satisfying conclusions A and B of Theorem II.

3. Plane curves. Necessary and sufficient conditions for planeness follow a

pattern similar to that which we encountered in the case of rectilinearity. The planeness of $\mathbf{x}(t)$ is expressed by writing

$$(\mathbf{a}\cdot\mathbf{x}(t)) = k \tag{3}$$

where $\mathbf{a}$ is the direction of the normal to the plane of the curve, and k is a scalar constant. We assume that $\mathbf{x}(t)$ is of class C^3. From section 2 it is clear that

$$\mathbf{x}'(t) \times \mathbf{x}''(t) \not\equiv 0$$

since the class of plane curves is here restricted to curves which are not (sectionally) straight lines. From (3),

$$(\mathbf{a}\cdot\mathbf{x}') \equiv (\mathbf{a}\cdot\mathbf{x}'') \equiv (\mathbf{a}\cdot\mathbf{x}''') \equiv 0. \tag{4}$$

Now we define

$$(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) = \begin{vmatrix} x_1' & x_1'' & x_1''' \\ x_2' & x_2'' & x_2''' \\ x_3' & x_3'' & x_3''' \end{vmatrix}.$$

Since $\mathbf{a}$ is not the null vector, equations (4) have a non-vanishing solution. It follows that, for all values of t, the determinant of these equations vanishes:

$$(\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}''') \equiv 0.$$

This necessary condition for planeness is expressed in

THEOREM IV. *If*:

1. $\mathbf{x}(t)$ *is of class* C^3,
2. $\mathbf{x}(t)$ *represents a plane curve,*

then:

A. $\mathbf{x}'(t)\times\mathbf{x}''(t)\not\equiv 0$,

B. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t))\equiv 0$.

That the converse of this theorem fails is apparent from the example

$$\mathbf{x} = (t^4, t, 0), \quad t \leqq 0; \qquad \mathbf{x} = (0, t, t^4), \qquad t > 0.$$

Although conclusions A and B above apply, this curve departs from planeness at the origin, where $\mathbf{x}'\times\mathbf{x}''$ vanishes (although $\mathbf{x}'(0)$ is non-zero). To avoid such exceptional behavior, we specify for the present that

$$\mathbf{x}'(t) \times \mathbf{x}''(t) \text{ never vanishes,} \tag{5}$$

as well as a differentiability of C^3 and conclusion B of Theorem IV. Applying the Lagrange identity [7, p. 51] to conclusion B, we have

$$(\mathbf{x}' \times \mathbf{x}''\cdot(\mathbf{x}' \times \mathbf{x}''') \times \mathbf{w}) = (\mathbf{x}'\cdot\mathbf{x}' \times \mathbf{x}''')(\mathbf{x}''\cdot\mathbf{w}) - (\mathbf{x}''\cdot\mathbf{x}' \times \mathbf{x}''')(\mathbf{x}'\cdot\mathbf{w}) \equiv 0$$

where $\mathbf{w}$ is an arbitrary or "dummy" vector. This can be rewritten

$$(\mathbf{x}' \times \mathbf{x}''\ \mathbf{x}' \times \mathbf{x}'''\ \mathbf{w}) = ((\mathbf{x}' \times \mathbf{x}'') \times (\mathbf{x}' \times \mathbf{x}''')\cdot\mathbf{w}) \equiv 0$$

which implies that

$$(\mathbf{x}' \times \mathbf{x}'') \times (\mathbf{x}' \times \mathbf{x}''') \equiv 0. \tag{6}$$

Next we introduce the transformation

$$\mathbf{y} = \mathbf{x}' \times \mathbf{x}''; \qquad \mathbf{y}' = \mathbf{x}' \times \mathbf{x}''' + \mathbf{x}'' \times \mathbf{x}'' = \mathbf{x}' \times \mathbf{x}'''$$

so that (6) becomes

$$\mathbf{y} \times \mathbf{y}' \equiv 0.$$

Thus, vector $\mathbf{y}$ is parallel to $\mathbf{y}'$, or

$$\mathbf{y}'(t) = k(t)\mathbf{y}(t)$$

in terms of scalar function $k(t)$. We have

$$\frac{d}{dt}\, y_i(t) = k(t)y_i(t); \qquad i = 1, 2, 3,$$

and, invoking (5) and proceeding as in the proof of Theorem III,

$$y_i(t) = a_i g(t); \qquad g(t) = e^{\int k(t)dt} \neq 0. \tag{7}$$

From the definition of $\mathbf{y}(t)$,

$$(\mathbf{y} \cdot \mathbf{x}') = (\mathbf{x}'\ \mathbf{x}''\ \mathbf{x}') \equiv 0,$$

and we use relation (7) to obtain

$$(\mathbf{a} \cdot \mathbf{x}'(t)) = \frac{1}{g(t)}\,(\mathbf{y}(t) \cdot \mathbf{x}'(t)) \equiv 0.$$

Integration gives

$$(\mathbf{a} \cdot \mathbf{x}(t)) = k,$$

so that, recalling (3), $\mathbf{x}(t)$ must be a plane curve, since it cannot be a straight line by Theorem II and condition (5). Sufficient conditions for planeness of space curves are expressed in

THEOREM V. *If*:

1. $\mathbf{x}(t)$ *is of class* C^3,
2. $\mathbf{x}'(t) \times \mathbf{x}''(t)$ *never vanishes*,
3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

4. Refinements. *The analytic case.* We have seen the exceptional behavior exhibited at the origin by the two examples which suggested the proofs of Theorems III and V. These two theorems respectively prohibit the vanishing of $\mathbf{x}'(t)$ and of $\mathbf{x}'(t) \times \mathbf{x}''(t)$ at any point. Yet this complete prohibition is unnecessarily drastic, as a pair of curves will show. The curve

$$\mathbf{x} = (0, t^3, 0)$$

is a straight line throughout, despite the vanishing of $\mathbf{x}'(0)$. Similarly,

$$\mathbf{x} = (0, t, t^4)$$

lies wholly in one plane, even though $\mathbf{x}'(0) \times \mathbf{x}''(0)$ vanishes. Graustein [4] typifies the approach of writers who discuss rectilinearity and planeness with the unnecessary strictness indicated by these two curves.

These remarks suggest that we attempt refinements of the conditions of Theorems III and V in order to admit as representatives of straight lines or plane curves many equations which are at present excluded. The basis for the first refinement will be the requirement that $\mathbf{x}(t)$ (*i.e.*, each of its component functions) be analytic throughout its domain.

We begin by taking the requirement of analyticity together with conclusions A and B of Theorem II as possible sufficient conditions for rectilinearity to replace those of Theorem III. Since $\mathbf{x}(t)$ is analytic, it follows that $\mathbf{x}'(t)$ is also analytic. A standard theorem in analysis states that the zeros of an analytic function are isolated on its domain unless the function vanishes identically [2]. This implies that $\mathbf{x}'(t)$ (*i.e.*, at least one of its components) never vanishes in some sufficiently small neighborhood I of a value t_0 of t which is not a zero of $\mathbf{x}'(t)$. The existence of such a t_0 follows from the assumed conclusion A of Theorem II. For values of t in I, $\mathbf{x}(t)$ is a straight line, since the conditions of Theorem III are fulfilled. Thus in I we can write

$$\mathbf{x}'(t) = \mathbf{a}f(t) \tag{8}$$

where $\mathbf{a}$ is the direction of the straight line represented by $\mathbf{x}(t)$ for t in I, and $f(t)$ is a scalar function.

$$x_i'(t) = a_i f(t); \qquad i = 1, 2, 3 \tag{8'}$$

gives (8) component-wise. If we assume that all the a_i are non-zero, it is clear that the functions

$$\frac{1}{a_i} x_i'(t); \qquad i = 1, 2, 3$$

are analytic in the entire domain of $\mathbf{x}(t)$. Furthermore, by (8'), these functions coincide in the finite interval I. Therefore, they coincide throughout the domain of $\mathbf{x}(t)$, by the identity theorem for analytic functions [6]. We may name this common expression $f(t)$. Thus

$$\frac{1}{a_i} x_i'(t) = f(t); \qquad i = 1, 2, 3$$

from which we obtain an expression of the form (8) for the entire domain of $\mathbf{x}(t)$, implying rectilinearity throughout.

If, in equation (8′), one of the a_i vanishes, say a_j, then still for the two remaining components, (8′) extends to all values of t as before. Since $x_j'(t)$ must vanish in I in this case, the identity theorem assures that, for all t,

$$x_j'(t) \equiv 0 = a_j f(t),$$

so that again equation (8) holds on the entire domain for all components. Further, if two of the a_i vanish in (8′), it is apparent from the identity theorem that the corresponding $x_i'(t)$ vanish for all t so that, interpreting geometrically, $\mathbf{x}'(t)$ has everywhere constant direction. Finally, not all three a_i can vanish since, as we have already observed, at least one component of $\mathbf{x}'(t)$ never vanishes in I. In every case $\mathbf{x}(t)$ is a straight line throughout, and we have established a refined expression of sufficient conditions for rectilinearity:

THEOREM IIIa. *If*:
1. $\mathbf{x}(t)$ *is analytic throughout its domain*,
2. $\mathbf{x}'(t) \not\equiv 0$,
3. $\mathbf{x}'(t) \times \mathbf{x}''(t) \equiv 0$,

then equation (1) *represents a straight line.*

The development of a first revised theorem concerning planeness of $\mathbf{x}(t)$ follows a similar plan. In addition to the analyticity of $\mathbf{x}(t)$, we adopt as hypotheses conclusions A and B of Theorem IV. The vector $\mathbf{x}'(t) \times \mathbf{x}''(t)$ plays a role similar to that of $\mathbf{x}'(t)$ in the foregoing proof. By an argument like that which led to equation (8), it appears that the conditions of Theorem V apply in some interval I of values of t, so that $\mathbf{x}(t)$ is a plane curve in this interval. The vector $\mathbf{x}'(t) \times \mathbf{x}''(t)$ becomes the normal to the plane of $\mathbf{x}(t)$ for t in I. For these values of t,

$$\mathbf{x}'(t) \times \mathbf{x}''(t) = \mathbf{b}g(t)$$

in terms of vector $\mathbf{b}$ and scalar function $g(t)$. This equation is extended over the entire domain of $\mathbf{x}(t)$ by the method applied to equation (8). Thus, $\mathbf{x}'(t) \times \mathbf{x}''(t)$ has everywhere constant direction. This implies that $\mathbf{x}(t)$ is restricted to a single plane, and our revision of sufficient conditions for planeness becomes

THEOREM Va. *If*:
1. $\mathbf{x}(t)$ *is analytic throughout its domain*,
2. $\mathbf{x}'(t) \times \mathbf{x}''(t) \not\equiv 0$,
3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

5. The non-analytic case. A final refinement of the conditions of Theorems III and V arises when we investigate wider classes of non-analytic expressions $\mathbf{x}(t)$ for which rectilinearity or planeness hold. For rectilinearity, we specify that $\mathbf{x}(t)$ is of class C^n, where $n > 1$, and that, at every point, at least one vector derivative of order less than or equal to n is different from zero. Conclusion **B** of Theorem II is also needed.

Under these hypotheses, if t_0 is an arbitrary value of t, there is a value of k for which

$$x_j^{(k)}(t_0) \neq 0; \qquad k \leqq n$$

where j indicates some particular component of $\mathbf{x}^{(k)}(t)$. We may suppose that $k>1$ and is the least value for which such a non-zero relation holds at t_0 in any component. The continuity of $\mathbf{x}^{(k)}(t)$ implies that, within some finite neighborhood I of t_0,

$$x_j^{(k)}(t) = f(t) \neq 0,$$

so that either $f(t)>0$ or $f(t)<0$ in I. The existence and uniqueness theorems [5] assure that this last equation has the solution for $x_j'(t)$ of the form [3]

$$x_j'(t) = \int_{t_0}^{t} dt \int_{t_0}^{t} dt \cdots \int_{t_0}^{t} f(t)dt, \tag{9}$$

involving $k-1$ integrations. All constants of integration vanish, owing to the relations

$$x_j'(t_0) = x_j'(t_0) = \cdots = x_j^{(k-1)}(t_0) = 0.$$

From the non-zero property of $f(t)$ we see that, at each step of integration, the integral in (9) cannot vanish in I except at t_0. This shows that the zeros of $\mathbf{x}'(t)$ and of its first $k-2$ derivatives are isolated points. Thus $\mathbf{x}(t)$ is at least sectionally straight. Specifically, in I, there is at worst a kink at t_0, since this is the only point where Theorem III breaks down.

It follows from the foregoing that, on the left of t_0 in I, $\mathbf{x}'(t)$ has the form $g(t)(a_1, a_2, a_3)$, where the a_i are the constant components of a unit vector. $g(t)$ is a scalar function. A similar expression involving a unit vector $\mathbf{b}$ holds on the right of t_0 in I. Successive differentiations of these expressions show that the first k derivatives of $\mathbf{x}(t)$ have the direction vectors $\mathbf{a}$ and $\mathbf{b}$ to the left and right of t_0, respectively. But any non-null vector $\mathbf{u}$ has its direction given by components $u_i/\sqrt{\mathbf{u}\cdot\mathbf{u}}$ so that

$$\frac{x_i^{(k)}(t)}{\sqrt{\mathbf{x}^{(k)}(t)\cdot\mathbf{x}^{(k)}(t)}} = a_i,\ t < t_0 \quad \text{and} \quad \frac{x_i^{(k)}(t)}{\sqrt{\mathbf{x}^{(k)}(t)\cdot\mathbf{x}^{(k)}(t)}} = b_i,\ t > t_0; \qquad i = 1, 2, 3.$$

By continuity and the non-zero condition on $\mathbf{x}^{(k)}(t)$ in I, the limits on the left and right at t_0 of the fraction appearing above are equal:

$$a_i = b_i; \qquad i = 1, 2, 3.$$

By this reasoning, we can extend the domain of direction vector $\mathbf{a}$ through all zeros of $\mathbf{x}'(t)$, so that $\mathbf{x}'(t)$ has constant direction throughout. We may state

THEOREM IIIb. *If*:

1. $\mathbf{x}(t)$ *is of class* C^n; $n>1$,
2. *at every point at least one vector derivative* $\mathbf{x}^{(k)}(t)$ *is different from zero*; $k \leqq n$,
3. $\mathbf{x}(t)\times\mathbf{x}''(t)\equiv 0$,

then equation (1) *represents a straight line.*

For planeness of $\mathbf{x}(t)$, the conditions are a differentiability of class C^n, where $n>2$, the existence at every point of some pair of vector derivatives $\mathbf{x}^{(k)}(t)$ and $\mathbf{x}^{(m)}(t)$ (where $k+m-1\leqq n$) whose outer product is different from zero, and conclusion B of Theorem IV. Thus for arbitrary t_0, if the conditions of Theorem V do not hold,

$$\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)\neq 0; \qquad k+m>3.$$

It is no restriction to specify that k is some least integer and m a next least integer for which this is true.

From the formula

$$(\mathbf{y}\times\mathbf{z})'=\mathbf{y}\times\mathbf{z}'+\mathbf{y}'\times\mathbf{z}$$

for the derivative of an outer product, the reader will see that the pth derivative of $\mathbf{x}'\times\mathbf{x}''$ is a sum of linear terms in $\mathbf{x}^{(u)}(t)\times\mathbf{x}^{(v)}(t)$ with positive integral coefficients and with

$$u+v=p+3; \qquad 1\leqq u<v.$$

Furthermore, all possible combinations of u and v satisfying this relation will appear in the derivative. The $(k+m-3)$rd derivative of $\mathbf{x}'\times\mathbf{x}''$ therefore has the form

$$K\mathbf{x}^{(k)}(t)\times\mathbf{x}^{(m)}(t)+\text{linear terms in }\mathbf{x}^{(r)}(t)\times\mathbf{x}^{(s)}(t); \qquad r+s=k+m, \tag{10}$$

where K is a positive integer. Now all the expressions $\mathbf{x}^{(r)}\times\mathbf{x}^{(s)}$ fall into one of two categories, as follows.

First, we may have $r<k$. Yet since we specified above that k is the least integer which appears in a non-zero outer product at t_0, it follows that all $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ in this category vanish.

Second, all remaining terms must have r and s between k and m. Speaking geometrically, if $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ is non-zero, then $\mathbf{x}^{(r)}(t_0)$ and $\mathbf{x}^{(s)}(t_0)$ are non-null vectors which are not parallel. In the case under consideration, this would imply $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(r)}(t_0)$ or $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ or both are different from zero. But any of these alternatives would contradict the hypothesis that m is the least value after k for which $\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)$ does not vanish.

We must admit that all the $\mathbf{x}^{(r)}(t_0)\times\mathbf{x}^{(s)}(t_0)$ vanish so that, at t_0, the derivative (10) reduces to the non-null vector $K\mathbf{x}^{(k)}(t_0)\times\mathbf{x}^{(m)}(t_0)$. By this fact and continuity, we have in a neighborhood I of t_0 for at least one jth component,

$$\frac{d^M(\mathbf{x}'(t) \times \mathbf{x}''(t)_j)}{dt^M} = F(t) \not\equiv 0; \qquad M = k + m - 3.$$

We may employ the same argument used on $x_j^{(k)}(t)$ in establishing equation (9) before. Again, all constants of integration vanish, since it follows from reasoning like that applied above to the Mth derivative of $\mathbf{x}' \times \mathbf{x}''$ that all derivatives of order less than M vanish at t_0. We obtain, after M integrations,

$$\mathbf{x}'(t) \times \mathbf{x}''(t)_j = \int_{t_0}^{t} dt \int_{t_0}^{t} dt \cdots \int_{t_0}^{t} F(t)dt.$$

Reasoning as before, the zeros of $\mathbf{x}' \times \mathbf{x}''$ and of its derivatives are isolated points, and Theorem V holds at least sectionally.

The rest is clear. We treat $\mathbf{x}' \times \mathbf{x}''$ as we did $\mathbf{x}'$ in the straight line case, establishing constant direction vectors on the left and right of t_0. Using continuity and the non-zero property of the Mth derivative in I, we prove equality of these vectors. $\mathbf{x}' \times \mathbf{x}''$ has everywhere constant direction and is normal to the plane of $\mathbf{x}(t)$. This final result is expressed in

THEOREM Vb. *If*:

1. $\mathbf{x}(t)$ *is of class* C^n; $n > 2$,

2. *at every point at least one pair of vector derivatives* $\mathbf{x}^{(k)}(t)$ *and* $\mathbf{x}^{(m)}(t)$ *has an outer product different from zero*; $k + m - 1 \leqq n$,

3. $(\mathbf{x}'(t)\mathbf{x}''(t)\mathbf{x}'''(t)) \equiv 0$,

then equation (1) *represents a plane curve.*

The two pairs of refined theorems admit to the classes of straight lines and plane curves many expressions which the original Theorems III and V exclude. The two curves discussed at the beginning of section 4 are examples. Furthermore, by admitting to consideration curves which possess singular points, our new hypotheses eliminate the unnecessary strictness in the traditional discussion of rectilinearity and planeness.

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THE EXPANSION OF A FUNCTION IN TERMS OF ITS VALUES AND DERIVATIVES AT SEVERAL POINTS

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In 1949 under the title *A Generalization of Taylor's Expansion*, Hummel and Seebeck* gave an interesting and useful expansion for $f(x)$ in terms of the derivatives at x, $f(a)$, and the derivatives at $x=a$. Their general theorem is

$$(1)\quad f(x) = f(a) + \sum_{k=1}^{\infty} \frac{(m+n-k)!}{(m+n)!} [C_{m,k} f^k(a) - (-1)^k C_{n,k} f^k(x)](x-a)^k + R.$$

where

$$R = \frac{(-1)^n m! n! (x-a)^{m+n+1}}{(m+n)!(m+n+1)!} f^{m+n+1}(\theta),$$

and θ lies between x and a. In this expansion the convention that a binomial coefficient $C_{m,k}$ vanishes for all values of k greater than m has been used, so that in the summation, the first term in the square bracket produces m terms whilst the second term in this bracket produces n terms.

In view of the particularly good approximations given by this theorem, it seemed to be of interest to express it, if possible, as a contour integral in the complex plane, and the result obtained is most interesting. The form of the appropriate contour integral is suggested by writing

$$f^k(a) = \frac{k!}{2\pi i} \int_C (t-a)^{-k-1} f(t)\,dt \text{ in the sum} \sum_{k=0}^{m} \frac{(m+n-k)!}{(m+n)!} C_{m,k} f^k(a)$$

whence, making use of the binomial expansion,

$$(1-u)^{-(n+1)} = \sum_{s=0}^{\infty} C_{n+s,s} u^s,$$

we obtain an integral of the form

$$(2)\qquad I(m,n) = \frac{1}{2\pi i} \int_C \frac{(x-a)^{m+n+1}}{(t-a)^{m+1}(t-x)^{n+1}} f(t)\,dt.$$

The process now becomes clear; The contour C must include the points x, a, and $f(t)$ must be regular within and on the contour. It then follows by Cauchy's theorem that the integral has a value equal to the sum of the residues of the integrand at the points $t=x$ and $t=a$, and clearly the residue at $t=x$ is expressible in terms of $f(x)$ and its first n derivatives, whilst the residue at $t=a$ is expressible in terms of $f(a)$ and its first m derivatives.

We now calculate the residue for $I(m, n)$ at the point $t=a$ in the usual way by putting $t=a+h$, the residue then being the coefficient of h^m in the expansion

* P. M. Hummel and C. L. Seebeck, Jr., this MONTHLY, vol. 56, 1949, pp. 243–247.

of

$$\frac{(x-a)^{m+n+1}}{(a-x+h)^{n+1}} f(a+h); \quad \text{hence}$$

the residue is equal to the coefficient of h^m in

$$(-1)^{m+1}\left(1+\frac{h}{a-x}\right)^{-(n+1)}(x-a)^m f(a+h)$$

which equals the coefficient of h^m in

$$(-1)^{m+1}(x-a)^m \sum_{s=0}^{\infty} C_{n+s,s}\left(\frac{h}{x-a}\right)^s f(a+h)$$

$$= \sum_{k=0}^{m} (-1)^{m+1}(x-a)^k C_{m+n-k,m-k}\,\frac{1}{k!} f^k(a)$$

$$= (-1)^{m+1}\frac{(m+n)!}{m!n!}\sum_{k=0}^{m}\frac{(m+n-k)!}{(m+n)!} C_{m,k} f^k(a)(x-a)^k.$$

Similarly, or by interchanging x, a and also m, n, the residue at $t=x$ becomes

$$(-1)^m \frac{(m+n)!}{m!n!}\sum_{k=0}^{n}\frac{(m+n-k)!}{(m+n)!} C_{n,k} f^k(a)(a-x)^k,$$

whence

$$\sum_{k=0}^{\infty}\frac{(m+n-k)!}{(m+n)!}\left[C_{m,k} f^k(a) - (-1)^k C_{n,k} f^k(x)\right](x-a)^k$$

$$= (-1)^{m+1}\frac{m!n!}{(m+n)!} I(m,n),$$

which is equivalent to the original theorem (1) with the remainder expressed as a contour integral.

Extensive generalization is now easy. If, for example, we wish to expand $f(x)$ in terms of $f(a)$ and its first $n-1$ derivatives and $f(b)$ and its first $m-1$ derivatives, we take as the appropriate contour integral

$$I(m,n) = \frac{1}{2\pi i}\int_C \frac{(x-a)^m(x-b)^n}{(t-a)^m(t-b)^n}\,\frac{f(t)}{(t-x)}\,dt. \tag{3}$$

The residue at $t=x$ is clearly $f(x)$, and the residues at $t=a$ and $t=b$ follow easily by the usual process. If we introduce a function

$$g_s^n(x) \equiv 1 + nx + \frac{n(n+1)}{1.2}x^2 + \cdots + \frac{n(n+1)\cdots(n+s-1)}{s!}x^s,$$

so that $g_s^n(x)$ is the sum of the first $s+1$ terms in the expansion of $(1-x)^{-n}$, and, for convenience also, write

$$\theta = \frac{x-a}{b-a}, \qquad \phi = \frac{x-b}{a-b},$$

then the corresponding expansion becomes,

$$f(x) = \phi^n \sum_{k=0}^{m-1} \frac{1}{k!} g_{m-k-1}^n(\theta) f^k(a)(x-a)^k + \theta^m \sum_{k=0}^{n-1} \frac{1}{k!} g_{n-k-1}^m(\phi) f^k(b)(x-b)^k + I(m, n). \tag{4}$$

In particular, setting $m=n$, we get the expansion,

$$f(x) = \sum_{k=0}^{n-1} \frac{1}{k!} \{\phi^n g_{n-k-1}^n(\theta) f^k(a)(x-a)^k + \theta^n g_{n-k-1}^n(\phi) f^k(b)(x-b)^k\} + I(n, n), \tag{5}$$

which as $n \to \infty$ gives an infinite series whose region of convergence is the interior of a Cassini oval.

As a further particular case we may extend the usual interpolation formula to give an approximation for $f(x)$ in terms of its values and the values of its first derivative at the points $x=a$, $x=b$, $x=c$. The appropriate contour integral will clearly be

$$I = \frac{1}{2\pi i} \int_C \frac{(x-a)^2(x-b)^2(x-c)^2}{(t-a)^2(t-b)^2(t-c)^2} \frac{f(t)}{(t-x)} dt. \tag{6}$$

The residues at the points $t=x, a, b, c$, are readily calculated, and quickly yield the formula,

$$f(x) = \sum \frac{(x-b)^2(x-c)^2}{(a-b)^2(a-c)^2} \left[f(a) + (x-a)f'(a) + 2\left\{\frac{a-x}{a-b} + \frac{a-x}{a-c}\right\} f(a) \right] + I, \tag{7}$$

in which the summation is taken over cyclic permutations of the letters a, b, c.

The complete generalization is now clear; for each point $x=a$, at which the expansion is to involve $f(a)$ and its first $m-1$ derivatives, we must place a factor $(t-a)^m$ in the denominator, and a corresponding factor $(x-a)^m$ in the numerator, of the integrand of the appropriate contour integral.

SYSTEMS OF DISTINCT REPRESENTATIVES

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1. Introduction. A certain dance is attended by n boys and n girls. Each boy has been previously introduced to exactly j girls, and each girl has been previously introduced to exactly k boys. Here k is necessarily an integer such that $1 \leqq k \leqq n$. No one desires to make any further introductions. Under these assumptions, will it always be possible for the boys and the girls to be paired with each other in such a way that no further introductions are necessary? The dance problem proposed here is actually one of considerable combinatorial significance. The purpose of our present paper is to analyze carefully a very general theorem of Philip Hall, which solves not only the dance problem, but a wide variety of related problems as well. The paper is in part expository, with certain simplifications in proof. However, Theorems 2.4 and 2.5 are new and give further insight into this type of combinatorial problem.

2. Systems of distinct representatives. Let $S_1, S_2, \cdots, S_n$ denote n subsets of a given set M. Let D be a set of n elements of M,

$$D = \{a_1, a_2, \cdots, a_n\},$$

such that the elements a_i are all distinct and such that a_i belongs to S_i for each $i = 1, \cdots, n$. Then D is said to be a system of distinct representatives for the subsets $S_1, S_2, \cdots, S_n$, (abbreviated by S.D.R.). Thus, for example, if our elements are integers and if $S_1 = \{1, 2, 3\}$, $S_2 = \{1, 2\}$, $S_3 = \{2, 3\}$, then a suitable D for S_1, S_2, S_3 is $D = \{1, 2, 3\}$. If the sets $S_1, S_2, \cdots, S_n$ have a S.D.R., then it is clear that any k of the sets must contain between them at least k elements. The converse proposition is the combinatorial theorem of Philip Hall [4]. Various elementary proofs of the P. Hall theorem are available (see for example, [1], [4], [5]). Theorem 2.1 which follows is actually a refinement of the P. Hall theorem and gives a lower bound for the number of S.D.R.'s. This bound was first obtained by M. Hall [3]. The proof of Theorem 2.1 given here is essentially the same as the Halmos and Vaughan proof of the P. Hall theorem [5]. Finally, it should be remarked that important extensions of P. Hall's theorem to the case in which the number n of the sets S_i is transfinite have been recently obtained by Everett and Whaples [1] and M. Hall [3]. Halmos and Vaughan [5] have also given an alternative proof of the transfinite form of the P. Hall theorem. We shall not be concerned with these generalizations here, but confine ourselves to the case in which n is finite. No restrictions, however, are imposed on the number of elements in any S_i, nor are any two of the S_i's necessarily even assumed to be distinct.

THEOREM 2.1. *Let $S_1, S_2, \cdots, S_n$ denote n subsets of a set M. For each $k = 1, 2, \cdots, n$, suppose that every k of these sets contain between them at least k distinct elements of M. Then there exists a* S.D.R. *for these subsets. Moreover, let r*

* The authors are indebted to the referee for his helpful suggestions.

be a fixed integer less than or equal to the minimal number of elements in each S_i. Then if $n \geq r$, there are at least $r!$ S.D.R.'s. If $n < r$, then there are at least $r!/(r-n)!$ S.D.R.'s.

The proof is by induction on n. For $n=1$, the result is trivial. Let $n>1$, and suppose that for each k, $1 \leq k < n$, every set of k S's contains at least $k+1$ distinct elements. Take the set S_1 and select from it any representative a_1. There are at least r choices for a_1. Now form the sets

$$\overline{S}_2 = S_2 - \{a_1\}, \cdots, \overline{S}_n = S_n - \{a_1\}.$$

Any k of the sets $\overline{S}_i$ contain between them at least k elements. By the induction hypothesis, if $r \leq n$, then $r-1 \leq n-1$, and there are at least $(r-1)!$ S.D.R.'s for the $\overline{S}_i$. If $n<r$, then $n-1<r-1$, and then there are at least $(r-1)!/(r-n)!$ S.D.R.'s. Hence there are at least $r!$ or $r!/(r-n)!$ S.D.R.'s.

Suppose then that for some k, $1 \leq k < n$, there are k S's, say $S_1, \cdots, S_k$, which contain exactly k distinct elements. Here we must have $k \geq r$, and by the induction hypothesis, these sets have at least $r!$ S.D.R.'s. Let $T = \{a_1, \cdots, a_k\}$ be any such S.D.R. Now form the sets

$$\overline{S}_{k+1} = S_{k+1} - T, \cdots, \overline{S}_n = S_n - T.$$

Any h of the $\overline{S}$ sets, $1 \leq h \leq n-k$, cannot contain fewer than h elements. For otherwise the S sets corresponding to these $\overline{S}$ sets along with $S_1, \cdots, S_k$ would contain fewer than $h+k$ elements. An application of the induction hypothesis to the $n-k$ sets $\overline{S}$ completes the proof.

P. Hall has also established the following important consequence of Theorem 2.1.

THEOREM 2.2. *Let $M = A_1 + \cdots + A_n$ be a partition of the set M into subsets $A_1, \cdots, A_n$ where $A_i \neq 0$ and* $A_i \cap A_j = 0$ for all $i \neq j$. Let $B_1, \cdots, B_m$ be m nonempty subsets of M. Suppose that for each k, $1 \leq k \leq m$, any k of the A's contain at most k of the B's. Then upon suitably renumbering m of the A's, $A_i \cap B_i \neq 0$ for $i = 1, \cdots, m$.*

Let S_i be the set of all A_j for which $A_j \cap B_i$ is not null. We assert that any k S's contain between them at least k A's. For if not, then k S's, say $S_1, \cdots, S_k$, contain at most $k-1$ A's. Thus the number of A's each of which intersects at least one of the sets $B_1, \cdots, B_k$ is at most $k-1$. But then the sets $B_1, \cdots, B_k$ are contained in the union of these A's, contrary to hypothesis. Hence by the preceding theorem, there exists a S.D.R. for $S_1, \cdots, S_m$ taken notationally as $A_1, \cdots, A_m$ such that $A_i \cap B_i \neq 0$ for $i = 1, \cdots, m$.

The theorem which follows is a special case of Theorem 2.2, and was first established by D. König [6].

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* The notation $A_i \cap A_j$ denotes the set of all elements in both A_i and A_j.

THEOREM 2.3. *Let M be a nonempty set with $M=A_1+\cdots+A_n=B_1+\cdots+B_n$, where $A_i\cap A_j=B_i\cap B_j=0$ for $i\neq j$ and where each A_i and B_j contains exactly m elements. Then the A's may be renumbered so that $A_i\cap B_i\neq 0$.*

The following two theorems describe circumstances under which certain specified elements may be selected as part of a S.D.R.

THEOREM 2.4. *Suppose that the sets $S_1, \cdots, S_n$ satisfy the conditions of Theorem* 2.1 *and suppose further that the elements $e_1, \cdots, e_k$ each occur among the sets $S_1, \cdots, S_n$ at least t times while each of the S_i contains at most t of the elements $e_1, \cdots, e_k$. Then there exists a* S.D.R. *containing $e_1, \cdots, e_k$.*

We shall call $e_1, \cdots, e_k$ *marginal elements.*

Let Σ be a S.D.R. which contains the highest possible number of marginal elements. We shall show that Σ contains all marginal elements. Suppose to the contrary that Σ contains exactly $e<k$ marginal elements.

The ordered set of sets $S_{j_1}, S_{j_2}, \cdots, S_{j_\alpha}$ will be called a *chain* if for $1\leqq\beta<\alpha$ the representative in Σ of S_{j_β} occurs in $S_{j_{\beta+1}}$. Note that by this definition one set S_j is a chain.

LEMMA. *Let i be a marginal element which does not occur in Σ. If i is in S_j then there is a chain connecting S_j with a set with a non-marginal representative.*

Deny this and consider all sets $S_{i_1}, \cdots, S_{i_\alpha}$ connected with S_j by a chain. Let $j_1, \cdots, j_\alpha$ be their representatives in Σ. Then $j_1, \cdots, j_\alpha$ are marginal elements and each occurs in the sets $S_{i_1}, \cdots, S_{i_\alpha}$ at least t times, for each set among $S_1, \cdots, S_n$ containing j_β is connected with S_{i_β}. Thus the sets $S_{i_1}, \cdots, S_{i_\alpha}$ contain together at least $\alpha t+1$ marginal elements whereas by assumption they cannot contain more than αt marginal elements. This proves the lemma.

Now let $S_j=S_{i_1}, S_{i_2}, \cdots, S_{i_\alpha}$ be the shortest chain such that S_{i_α} has a non-marginal representative β. Then $S_j, \cdots, S_{i_{\alpha-1}}$ have marginal representatives in Σ. We construct a system Σ' of representatives of $S_1, \cdots, S_n$ as follows. Let S_j be represented by i. Let S_{i_k} for $1<k\leqq\alpha$ be represented by the representative of $S_{i_{k-1}}$ in Σ. Let all other sets be represented by their representatives in Σ. Clearly Σ' is a S.D.R. and contains $e+1$ marginal elements contradicting the significance of e and Σ.

THEOREM 2.5. *Let $S_1, \cdots, S_n$ be sets satisfying the conditions of Theorem* 2.1 *and assume that $e_1, \cdots, e_k$ each occur among the sets $S_1, \cdots, S_n$ at least t times, while each of the sets S_i contains at most t elements. Let α be any element. Then there exists a* S.D.R. *containing $\alpha, e_1, \cdots, e_k$.*

The proof of Theorem 2.5 is analogous to that of Theorem 2.4. We start with a S.D.R. containing $e_1, \cdots, e_k$ and let α play the role of i in the preceding proof.

3. Applications. We will now consider the applications of the preceding combinatorial theorems to various mathematical investigations. The first application is due to König, and concerns matrices whose elements are composed

only of the integers 0 and 1 [6]. A matrix with a single entry of 1 in each row and in each column and with all other entries 0 is called a permutation matrix.

THEOREM 3.1. *Let A be an n by n matrix of zeros and ones, with exactly k ones in each row and in each column. Then $A=L_1+\cdots+L_k$, where the L_i are permutation matrices.*

Let the ith row of A have ones in columns $i_1, i_2, \cdots, i_k$, and zeros elsewhere. In this way, we may form n sets $S_i=\{i_1, i_2, \cdots, i_k\}$, each containing k integers from among the integers $1, 2, \cdots, n$. Any r of these sets contain between them at least r distinct elements. For if r sets contained at most $r-1$ elements, then in the corresponding rows of the matrix A there would be at most $k(r-1)$ ones, whereas there are kr ones. Thus the sets S_i fulfill the hypothesis of Theorem 2.1, and a S.D.R. gives rise to a permutation matrix L_1. We may proceed similarly with the matrix $A-L_1$, and Theorem 3.1 then follows by induction. Incidentally, Theorem 3.1 gives a solution to the dance problem described in the introduction. The permutation matrix L_1 gives a desired pairing of previously introduced boys and girls.

Another application of interest arises naturally in the study of symmetrical balanced incomplete block designs. A complete description of such designs may be found in [7]. These designs may be converted into Youden squares [10], and this fact is also a direct consequence of Theorem 2.1.

The next application which we will consider concerns the cosets of a finite group.

THEOREM 3.2. *Let G be a finite group and let H and K be two subgroups of order m. Let $G=Hx_1+\cdots+Hx_r$ be a right coset decomposition for H and let $G=y_1K+\cdots+y_rK$ be a left coset decomposition for K. Then it is possible to choose $x_1, \cdots, x_r$ so that $G=Hx_1+\cdots+Hx_r=x_1K+\cdots+x_rK$.*

From the elementary properties of cosets, it is clear that Theorem 3.2 is a direct consequence of Theorem 2.3. The theorem is frequently stated for the case in which $H=K$. Thus the left and right cosets of a subgroup H have a common system of representatives.

Another group-theoretic application of interest was developed by R. Rado, and is specifically concerned with the symmetric group of order $n!$ [8].

The final application which we will consider concerns the subject of latin rectangles. A latin rectangle of order r by s based upon the integers $1, 2, \cdots, n$ is defined as an array of r rows and s columns formed from the integers $1, 2, \cdots, n$ in such a way that the integers in each row and in each column are distinct. The latin rectangle is said to be extendible to an n by n latin square provided that it is possible to adjoin $n-s$ columns and $n-r$ rows in such a way that the resulting array is an n by n latin square. The following theorem concerning the extendibility of latin rectangles to n by n latin squares was first derived for the case of r by n latin rectangles [2] and then extended to the case of r by s latin rectangles [9].

THEOREM 3.3. *Let T be an r by s latin rectangle based upon the integers 1, 2, $\cdots$, n. Let $N(i)$ equal the number of times that the integer i occurs in T. A necessary and sufficient condition in order that T may be extended to an n by n latin square is that for each $i=1, 2, \cdots, n$, $N(i) \geqq r+s-n$.*

Let S_i denote the set of the $n-s$ integers which are not in the ith row of T. Let $M(i)$ be the number of times that i occurs among the sets $S_1, S_2, \cdots, S_r$. If T is extendible to a latin square, then i cannot occur among the sets $S_1, S_2, \cdots, S_r$ more than $n-s$ times, whence $M(i) \leqq n-s$. But $N(i)+M(i)=r$, whence $N(i) \geqq r+s-n$. Thus the condition is a necessary one.

To prove the sufficiency we note that by hypothesis, $M(i) \leqq n-s$. Then any k S's contain between them at least k distinct elements. For if they contained at most $k-1$ distinct elements, then the sum of the numbers of elements in these k sets would be at most $(k-1)(n-s)$, whereas it is $k(n-s)$. Thus by Theorem 2.4 the sets $S_1, S_2, \cdots, S_r$ have a S.D.R. which includes all i for which $N(i)=r+s-n$. We add this S.D.R. as $(s+1)$st column and the resulting r by $s+1$ latin rectangle satisfies again the conditions of Theorem 3.3. One may now repeat the process, selecting S.D.R.'s by means of Theorem 2.4 until a r by n latin rectangle is obtained. For the r by n latin rectangle, $N(i)=r \geqq r+s-n$, since $s-n \leqq 0$. Hence the same argument may be applied to its n by r transpose to yield the transpose of the desired n by n rectangle.

As a final direct consequence of Theorem 2.1, we state the theorem of M. Hall, which gives a lower bound for the number of r by n latin rectangles [3].

THEOREM 3.4. *There are at least $n!\,(n-1)!\cdots(n-r+1)!$ r by n latin rectangles, and hence at least $n!\,(n-1)!\cdots(1)!$ n by n latin squares.*

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MATHEMATICAL NOTES

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ON AN INEQUALITY DUE TO WEINBERGER

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In a recent issue of the Proceedings of the National Academy of Sciences, July, 1942, H. F. Weinberger proved the following interesting inequality

$$\sum_{n=1}^{k} (-1)^{n+1} a_n^r \geqq \left(\sum_{n=1}^{k} (-1)^{n+1} a_n \right)^r \tag{1}$$

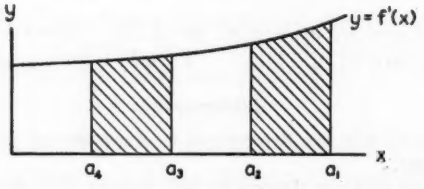
provided that $r \geqq 1$ and

$$a_1 \geqq a_2 \geqq \cdots \geqq a_n \geqq 0. \tag{2}$$

The purpose of this note is to present a generalization of (1), namely

$$\sum_{n=1}^{k} (-1)^{n+1} f(a_n) \geqq f\left(\sum_{n=1}^{k} (-1)^{n+1} a_n \right) \tag{3}$$

which holds provided that (2) is valid and that $f(0), f'(0) \geqq 0$, and $f'(x)$ is monotone increasing. The proof is immediate upon referring to the graph



and comparing areas. It is clear that equality results only if $a_2 = a_3$, $a_4 = a_5$, *etc.*

ON THE CONVERGENCE OF ALTERNATING DOUBLE SERIES

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1. Introduction. Several years ago a new definition of convergence of double series, σ-convergence, was proposed independently by Sheffer [4] and Amerio [1]. This new method provides a theory more nearly analogous to that of simple series than is obtained by Pringsheim convergence. Somewhat later Wilansky [5] compared σ-convergence with regular convergence and showed that the latter has most of the desirable properties of the former but that the two definitions are not equivalent, σ-convergence being the more stringent requirement. It is our purpose to give a necessary and sufficient condition for the σ-convergence of a monotonic alternating double series and to compare the two definitions

of convergence in a special case.

2. Definitions. For the definitions of Pringsheim and σ-convergence see [4] or [5].

A double series is said to be regularly convergent if it is Pringsheim convergent and summable by rows and columns.

Consider the double series $\sum_{i,j=1}^{\infty} a_{ij}$. The partial sums $D_k=\sum_{i+j=k} a_{ij}$ will be called the diagonals of the series. We shall also wish to consider the triangular partial sums $T_k=\sum_{i+j\leq k} a_{ij}$. The symbols D_k, T_k, σ, *etc.*, will be used to denote both certain regions of the double array and also the sums of the elements contained in these regions.

A series will be said to be *diagonally summable* if $\sum_{n=2}^{\infty} D_n$ converges.

We shall call a double series *alternating* if each row and column is an alternating simple series. The alternating series $\sum a_{ij}$ will be called *monotonic* if $|a_{ij}| \leqq |a_{mn}|$ for $i \geqq m$, $j \geqq n$.

3. Conditions for convergence.

THEOREM. *A necessary and sufficient condition for the σ-convergence of a monotonic alternating double series $\sum_{i,j=1}^{\infty} a_{ij}$ is diagonal summability.*

Necessity: This follows immediately from the fact that the triangular sums T_k are σ-sums.

Sufficiency: Since $\sum_{n=2}^{\infty} D_n$ converges, for every $\epsilon>0$ there exists an integer p such that

$$\left|\sum_{n=q}^{q+r} D_n\right| < \epsilon/4 \qquad \text{for } q \geqq p,\ r \geqq 0. \tag{1}$$

Let σ' be any σ-sum (p, p). Then σ' will contain T_p. Let T_s be the largest triangular sum contained in σ'. Hence there must be at least one element a_{mn} in D_{s+1} which is not in σ'. If there are several such elements, choose a_{mn} as the one of them with minimum n. Let $S_1=T_s-T_p$. Thus $\sigma'-T_s$ is divided into two portions—a region S_2 in the first $n-1$ columns only, and a region S_3 in the first $m-1$ rows only. (If $n=1$, S_2 will not exist; if $n=s$, S_3 will not exist; and if σ' is a triangular sum, neither S_2 nor S_3 will exist. In these cases the following reasoning holds *a fortiori*.) S_3 may consist of several disconnected polygons.

By (1), $|S_1| <\epsilon/4$.

By the theorem of Leibniz on monotonic alternating simple series, the sum of the portion of each column in S_2 is not greater in magnitude than the term of D_{s+1} in that column and is of the same sign. Similarly, the sum of the portion of each row in S_3 is not greater in magnitude than the term of D_{s+1} in that row and is of the same sign. Hence,

$$|S_2| + |S_3| \leqq |D_{s+1}| < \epsilon/4, \quad \text{by (1)}.$$

Therefore,

$$|\sigma' - T_p| \leqq |S_1| + |S_2| + |S_3| < \epsilon/4 + \epsilon/4 = \epsilon/2.$$

Now let σ' and σ'' be any two σ-sums (p, p). Then

$$|\sigma' - \sigma''| \leqq |\sigma' - T_p| + |\sigma'' - T_p| < \epsilon/2 + \epsilon/2 = \epsilon,$$

and the series is σ-convergent.

4. An example. It is well-known that for simple series $\sum_{i=1}^{\infty}(-1)^{i+1}i^{-n}$ is convergent for $n>0$ and is absolutely convergent for $n>1$. The analogues of these results for regular and σ-convergence provide an interesting comparison of the two definitions.

We consider the series $\sum_{i,j=1}^{\infty}(-1)^{i+j}(i+j-1)^{-n}$. Since $\int_1^{\infty}\int_1^{\infty}(x+y-1)^{-n}\,dxdy$ converges for $n>2$, the above series is absolutely Pringsheim convergent for $n>2$. (See [2], p. 86.) It is shown in [4] that absolute Pringsheim convergence and absolute σ-convergence are equivalent.

Pringsheim has shown [3] that a monotonic alternating double series is Pringsheim convergent if

$$|a_{ij} + a_{i+1,j}| \geqq |a_{i,j+1} + a_{i+1,j+1}|$$

for all i and j. The series $\sum(-1)^{i+j}(i+j-1)^{-n}$ satisfies this condition for $n>0$, since $x^{-n}-(x+1)^{-n}$ is a decreasing function of x for $x\geqq 1$, $n>0$. Hence, the above double series is regularly convergent if $n>0$, and it is easily seen that the series diverges for $n\leqq 0$.

However, if we set $n=1$ and attempt to sum the series by diagonals, we obtain $1-1+1-1+\cdots$; so this regularly convergent series is not σ-convergent. The theorem of the last section shows that the series is σ-convergent if and only if $n>1$.

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A NOTE ON A THEOREM OF PARKER

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Recently W. V. Parker [2] proved the following theorem:

Let A be an $n\times m$ matrix of rank $r<n$ and let C be an $m\times n$ matrix such that $ACA=kA$ (k a scalar). If B is an $m\times n$ matrix, the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $A(B+C)$ is $x^{n-r}\phi(x-k)=0$.

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It may be remarked that this result can be generalized somewhat in the same manner that W. T. Reid [3] generalized an earlier theorem of Parker [1].

THEOREM 1. *Let A be an $n\times m$ matrix of rank r and D an $n\times n$ matrix such that $DA=kA$. If B is an $m\times n$ matrix, then the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and the characteristic equation of $AB+D$ is $g(x)\phi(x-k)=0$, where $g(x)$ is a polynomial of degree $n-r$.*

Proof. There exist nonsingular matrices P and Q such that

$$PAQ=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}.$$

If we set

$$PDP^{-1}=\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix},$$

we get

$$PDAQ=PDP^{-1}PAQ=\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}=\begin{pmatrix} D_1 & 0 \\ D_3 & 0 \end{pmatrix}. \tag{1}$$

But $DA=kA$, hence

$$PDAQ=kPAQ=\begin{pmatrix} kI_r & 0 \\ 0 & 0 \end{pmatrix}. \tag{2}$$

From (1) and (2) we get

$$D_1=kI_r \quad \text{and} \quad D_3=0;$$

hence

$$PDP^{-1}=\begin{pmatrix} D_1 & D_2 \\ D_3 & D_4 \end{pmatrix}=\begin{pmatrix} kI_r & D_2 \\ 0 & D_4 \end{pmatrix}. \tag{3}$$

Now we apply the similarity transformation P to AB and $AB+D$ and set

$$Q^{-1}BP^{-1}=\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}.$$

Now

$$PABP^{-1}=PAQQ^{-1}BP^{-1}=\begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}=\begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} \tag{4}$$

and by (3) and (4)

$$P(AB+D)P^{-1} = PABP^{-1} = \begin{pmatrix} B_1 & B_2 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} kI_r & D_2 \\ 0 & D_4 \end{pmatrix} = \begin{pmatrix} B_1 + kI_r & B_2 + D_2 \\ 0 & D_4 \end{pmatrix}. \tag{5}$$

From (4) and (5) we conclude that the characteristic equation of AB is $x^{n-r}\phi(x)=0$ and that of $AB+D$ is $g(x)\phi(x-k)=0$ where $\phi(x)=0$ is the characteristic equation of B_1 and $g(x)=0$ the characteristic equation of D_4.

COROLLARY 1. *If D is of rank r, and $k\neq 0$, then $g(x)=x^{n-r}$.*

Proof. From (3) we see that $D_4=0$ since the rank of D is equal to the sum of the ranks of kI_r and D_4 and the rank of kI_r is r. From (5) we see that

$$g(x) = x^{n-r}.$$

COROLLARY 2. *If A is nonsingular, then $D=kI$.*

In the original form Reid's theorem cannot be generalized. His result shows up in

COROLLARY 3. *If D is nilpotent and $DA=kA$, then $k=0$.*

Proof. D is nilpotent if and only if PDP^{-1} is. But from (3) we see that PDP^{-1} is reduced; hence each main diagonal factor must be nilpotent. This implies

$$(kI_r)^t = 0,$$

which is possible only if $k=0$.

Reid's result may be obtained by observing that D_4 is nilpotent; hence it has as characteristic equation $x^{n-r}=0$.

By exactly the same method Parker [2] used we may prove:

THEOREM 2: *If $D^2=D$ and $k\neq 0$ (we may assume it to be 1), then the minimum functions of AB, say $g(x)$, and $AB+D$, say $h(x)$, must satisfy one of the four relations*

$$h(x) = g(x-1) \qquad h(x) = xg(x-1)$$
$$(x-1)h(x) = g(x-1) \qquad (x-1)h(x) = xg(x-1).$$

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THE NUMBER OF CYCLES ASSOCIATED WITH THE ELEMENTS OF A PERMUTATION GROUP

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Gontcharoff [1] has given several moment generating functions for the number of cycles associated with the elements of the permutation group on N symbols. Recently the author rediscovered the factorial moment generating function for the above problem previously given by Gontcharoff. Since the method of derivation is quite different from that given by Gontcharoff, it may be worth reproducing here.

For each positive integer N, we let G_N denote the group of $N!$ permutations on N symbols (which we may refer to as being cards numbered from 1 to N for convenience). As is well known, each of these permutations may be written as a product of disjoint cycles of lengths $t_1, t_2, \cdots, t_m$ say, and these lengths will satisfy the relation $\sum t_i = N$. We call the permutation $(1, 2, \cdots, N)$ the standard permutation.

The factorial moment generating function found by the author is based on the form

$$M_N = (x + N - 1)^{(N)} \tag{1}$$

where

$$y^{(n)} = y(y-1)\cdots(y-n+1). \tag{2}$$

The form (1) was found by empirical methods; the coefficient of x^p in the expansion of M_N is the number of permutations on N symbols (or elements of G_N) which have p cycles associated with them. This may be trivially verified for $N=1$, for then $M_1 = x$, and there is one cycle with unit length.

The correctness of (1) for the general case may be proved by induction. Assume that $(K+1)$ is the smallest positive integer N for which form (1) fails to provide a correct description of the number of cycles according to the above conventions. Then, by this hypothesis, M_K does provide a correct description for the elements of G_K.

Consider the groups G_{K+1} and G_K. For each of the fixed positions $1, 2, 3, \cdots, K+1$, there are $K!$ elements of G_{K+1} with card $(K+1)$ in that fixed position. We consider the effect of eliminating the card numbered $(K+1)$.

If the card numbered $(K+1)$ is in its standard position, then it of itself constitutes a cycle of length one. The deletion of this card reduces the number of cycles by unity. The remaining K cards are so distributed as to form an element of G_K. Since the form M_K describes the group G_K, the form xM_K (where the factor x adds one to the number of cycles associated with each element of the group G_K) describes the subset of G_{K+1} which has the card $(K+1)$ in its standard position.

If the card numbered $(K+1)$ is not in its standard position, we delete this card and think of the card in position $(K+1)$ as being moved to the position of

the deleted card. This procedure reduces the length of some one cycle associated with an element by unity. Now consider the subset of G_{K+1} where card $(K+1)$ is in fixed position L. By this process, each element of G_{K+1} has the same number of cycles as the reduced element after the deletion; the reduced elements are just the elements of the group G_K. Indeed, this procedure describes a one-to-one mapping of the subset of G_{K+1} where card $(K+1)$ is in fixed position L onto the group G_K. Hence the form M_K describes the number of cycles for this subset of G_{K+1}. Similar one-to-one mappings exist for each fixed position $L = 1, 2, 3, \cdots, K$. Hence there is a combined K to 1 mapping of the subset of G_{K+1} where card $(K+1)$ is not in its standard position onto the group G_K. Therefore the form KM_K describes this subset of G_{K+1}.

But card $(K+1)$ either is or is not in its standard position, and hence

$$xM_K + KM_K = (x + K)M_K = M_{K+1} \tag{3}$$

does describe the entire group G_{K+1}. Therefore there is no smallest positive integer which fails to provide a correct description, *i.e.*, the form M_N affords a correct description of the number of cycles associated with the elements of G_N for all positive integers N.

To reduce to a probability distribution, we divide M_N by $N!$, the number of elements in G_N.

To find the mean of the number of cycles p, we note that the arithmetical steps in evaluating

$$\frac{1}{N!}\left[\frac{dM_N}{dx}\right]_{x=1}$$

are just the steps used in calculating the mean from the definition,

$$\text{mean} = \frac{1}{N!}\sum pf_p$$

where f_p is the number of elements associated with p cycles. Hence we have

$$\begin{aligned}\text{mean} = \mu_1 &= \frac{1}{N!}\left[\frac{dM_N}{dx}\right]_{x=1} \\ &= \frac{1}{N!}\left[\frac{d}{dx}\{(x + N - 1)(x + N - 2)\cdots(x + 1)x\}\right]_{x=1} \\ &= 1 + 1/2 + 1/3 + \cdots + 1/N.\end{aligned}$$

For large N,

$$\mu_1 \simeq \log N + C$$

where C is Euler's constant, $C \simeq 0.577$.

Likewise the variance may be computed from

$$\text{Variance} = \frac{1}{N!}\left[\frac{d^2M_N}{dx^2}\right]_{x=1} + \mu_1 - (\mu_1)^2$$

$$= 1 + 1/2 + 1/3 + \cdots + 1/N - \left(1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{N^2}\right).$$

For large N,

$$\text{Variance} \simeq \log N + C - \frac{\pi^2}{6}.$$

Statistics involving higher order moments are obtainable from general formulas associated with the factorial moment generating function [2].

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CLASSROOM NOTES

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ON BLISS' SUBSTITUTE FOR DUHAMEL'S PRINCIPLE

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The so-called principle of Duhamel has been criticized and revised by W. F. Osgood [1] and R. L. Moore [2]. Then G. A. Bliss [3] replaced it by a much simpler theorem:

Let F be a continuous function in the closed, bounded, Jordan measurable region V of a Euclidean m-dimensional space. If V is divided into finitely many disjoint, Jordan measurable subregions with maximum diameter less than δ and with volumes denoted by ΔV_k $(k=1, 2, \cdots, n)$ and if in each of these subregions three points p_k, p_k', p_k'' are chosen, then

$$(1) \qquad \lim_{\delta \to 0} \sum_{k=1}^{n} F(p_k, p_k', p_k'') \cdot \Delta V_k = \lim_{\delta \to 0} \sum_{k=1}^{n} F(p_k, p_k, p_k) \cdot \Delta V_k = \int_V F dV.$$

The analogue holds if the number of points selected in each subregion were two or more than three.

By using the property of uniform continuity of F in V, the proof of Bliss'

theorem is obvious.

For the usual applications it is sufficient to have F in the form of a product of continuous functions [4]. For instance, let F be the product of three continuous functions, $F(p)=f(p)\cdot g(p)\cdot h(p)$ and replace (1) by

$$\lim_{\delta\to 0}\sum_{k=1}^{n} f(p_k)\cdot g(p_k')\cdot h(p_k'')\cdot \Delta V_k = \lim_{\delta\to 0}\sum_{k=1}^{n} f(p_k)\cdot g(p_k)\cdot h(p_k)\cdot \Delta V_k = \int_V f\cdot g\cdot h dV. \tag{2}$$

But then one can immediately generalize this statement by considering integrable functions instead of continuous functions. The proof for this generalization is very simple and has the advantage of avoiding the notion of uniform continuity entirely.

So, using the above notations, we have the following theorem concerning $F=f\cdot g\cdot h$:

THEOREM. *If f is bounded and g, h, and $f\cdot g\cdot h$ are (Riemann) integrable in V, then* (2) *holds.*

Proof. We have $|f|, |h|, |g| \leqq C$ in V and designate by $o_k(g)$ the oscillation of g (*i.e.*, l.u.b. g − g.l.b. g) in the subregion whose index is k; the meaning of $o_k(h)$ is analogous. Then

$$\begin{aligned}\sum_{k=1}^{n} f(p_k)\cdot g(p_k')\cdot h(p_k'')\Delta V_k &= \sum_{k=1}^{n} f(p_k)\cdot g(p_k)\cdot h(p_k)\Delta V_k \\ &+ \sum_{k=1}^{n} f(p_k)\cdot h(p_k)\cdot (g(p_k') - g(p_k))\Delta V_k \\ &+ \sum_{k=1}^{n} f(p_k)\cdot g(p_k')\cdot (h(p_k'') - h(p_k))\Delta V_k.\end{aligned} \tag{3}$$

But

$$\left|\sum_{k=1}^{n} f(p_k)\cdot h(p_k)\cdot (g(p_k') - g(p_k))\Delta V_k + \sum_{k=1}^{n} f(p_k)\cdot g(p_k')\cdot (h(p_k'') - h(p_k))\Delta V_k\right| \leqq C^2\cdot\left[\sum_{k=1}^{n} o_k(g)\Delta V_k + \sum_{k=1}^{n} o_k(h)\Delta V_k\right]. \tag{4}$$

The right hand term of (4) tends to 0 as $\delta\to 0$ by the classical condition for the integrability of g and h. Hence (2) follows.

All is analogous if F consists of only two factors or of more than three factors and the corresponding number of points p_k, p_k', $\cdots$ is chosen in each subregion of V. In particular, in the case of two factors of F, $F=f\cdot g$, and two points p_k, p_k' the above proof becomes even much simpler: there one may set $h=1$, and

hence the last sum on the right hand side of (3) does not occur.

The generalization to Riemann-Stieltjes integrals is immediate.

References

1. W. F. Osgood, Annals of Math., 2, vol. 4, 1903, pp. 161–178.
2. R. L. Moore, Annals of Math., 2, vol. 13, 1912, pp. 161–166. Another formulation of a similar type is stated, *e.g.*, in G. E. F. Sherwood and A. E. Taylor, Calculus (revised edition), New York, 1946, pp. 317–318.
3. G. A. Bliss, A substitute for Duhamel's theorem, Annals of Math., 2, vol. 16, 1914–15, pp. 45–49.
4. See H. B. Fine, Calculus, New York, 1927, pp. 163–164; G. E. F. Sherwood and A. E. Taylor, *loc. cit.* [2], Corollary, pp. 318 and 532; L. L. Smail, Calculus, New York, 1949, pp. 129–130; and other textbooks on Calculus. L. L. Smail, *loc. cit.*, called this case the "theorem of Bliss."

ON A NOWHERE-DENSE SET

ALBERT WILANSKY, Lehigh University

Early in the introduction of Lebesgue measure the natural question arises whether a nowhere-dense set must have measure 0, even whether it must be countable. Cantor's ternary set answers the second question in the negative, and may be modified to answer the first.

If the lecturer does not wish to interrupt the continuity of his presentation he may present the following brief example:

Let I be the closed interval $[0, 1]$. Fix a number t, $0<t<1$. Let $G(t)$ be an open set of measure less than or equal to t, including the set of rational numbers in I. (One has proved, earlier, that a countable set, and, in particular, the set of rational numbers, has measure 0.) Let F be the complement of $G(t)$ with respect to I.

THEOREM. *F is a closed, nowhere-dense (non-countable) set of measure greater than or equal to $1-t$.*

Thus F has some of the properties of the Cantor set. (If one stops to prove that every set of positive measure has a non-measurable subset, this yields the existence of a nowhere-dense non-measurable set.) Notice that the measure of F may be arbitrarily near 1. Notice also that the complement of $\bigcap_{n=1}^{\infty} G(1/n)$ is the well-known example of a set of first category which has measure 1.

AN ELEMENTARY DERIVATION OF THE FORMULAE FOR P_m AND $P_{[m]}$

J. E. FREUND, Alfred University

1. Introduction. The formulae for the probabilities of the realization of m (or at least m) among N events are usually presented by assuming the results and demonstrating that the proper points of the sample space contribute to the expressions while all others do not.* Since this method of proof is a verification

* W. Feller, An Introduction to Probability Theory and Its Applications, New York, 1950, p. 61.

rather than a derivation, it is the purpose of this paper to show how one can derive the formulae directly and in a, pedagogically speaking, more satisfactory manner.

Using the customary symbolism, p_i stands for the probability of the realization of event i, $p_{i,j}$ for the probability of the simultaneous realization of events i and j, *etc.* Also, the sums S_k are defined as

$$S_1 = \sum p_i, \quad S_2 = \sum p_{i,j}, \quad S_3 = \sum p_{i,j,k}, \quad etc.,$$

the probabilities being summed for all $1 \leqq i < j < k < \cdots \leqq N$. Finally, P_m stands for the probability of the realization of at least m among N events, while $P_{[m]}$ stands for the probability of the realization of exactly m among N events.

2. A useful identity. The derivation of the formulae for P_m and $P_{[m]}$ given below makes use of the following identity:

$$1 - \sum_{i=0}^{k-1} (-1)^i \binom{m-1+i}{i}\binom{m+k}{m+i} = (-1)^k \binom{m-1+k}{k} \tag{1}$$

which can also be written as

$$\sum_{i=0}^{k} (-1)^i \binom{m-1+i}{i}\binom{m+k}{m+i} = 1. \tag{2}$$

Beginning with the well-known identity†

$$(-1)^j \binom{a-1+j}{j} = \binom{-a}{j},$$

one can write

$$\sum_{i=0}^{k} (-1)^i \binom{m-1+i}{i}\binom{m+k}{m+i} = \sum_{i=0}^{k} \binom{-m}{i}\binom{m+k}{m+i} = \sum_{i=0}^{k} \binom{-m}{i}\binom{m+k}{k-i},$$

and it can easily be seen that the summation of the right-hand side of the equation is equal to 1 since it represents the coefficient of x^k in the expansion of

$$(1+x)^{-m}(1+x)^{m+k} = (1+x)^k.$$

This completes the proof of identity (1).

3. The derivation of the formula for P_m. If more than m of the N events cannot possibly occur simultaneously, it is evident that both P_m and $P_{[m]}$ are equal to S_m. If, on the other hand, it is possible that more than m events occur simultaneously, one has to consider duplications like, *e.g.*, for $m=1$ when the probability $p_{i,j}$ is counted twice, once in p_i and once in p_j. Since compound events

† *Ibid.*, p. 47.

are significant in this problem only for the number of events involved, not for the particular combination of events, the probabilities of compound events may be grouped into the sums S_k. Therefore, since all cases under consideration are covered by events whose probabilities occur in the sums S_m to S_N, the probability P_m must be of the form

$$P_m = \sum_{i=0}^{N-m} a_{m+i} S_{m+i}. \tag{3}$$

Since each case in which exactly m events are realized is contained once in S_m and in none of the later terms, it follows that $a_m = 1$. The next coefficient is equal to

$$a_{m+1} = 1 - \binom{m+1}{m}$$

because each case in which exactly $m+1$ events are realized is contained once in S_{m+1}, it is not contained in any of the later terms, and it has already been taken

$$\binom{m+1}{m}$$

times in S_m. Using identity (1), this coefficient can also be written as

$$a_{m+1} = -\binom{m}{1}. \tag{4}$$

Similarly, each case in which exactly $m+2$ events are realized must be taken once in S_{m+2}, minus the number of times that it has already been taken in $a_m S_m$ and $a_{m+1} S_{m+1}$. This gives

$$a_{m+2} = 1 - a_m \binom{m+2}{m} - a_{m+1} \binom{m+2}{m+1},$$

and using previous results and identity (1), one has

$$a_{m+2} = 1 - \binom{m+2}{m} + \binom{m}{1}\binom{m+2}{m+1} = \binom{m+1}{2}. \tag{5}$$

As further calculations will show that

$$a_{m+3} = -\binom{m+2}{3}$$

and

$$a_{m+4} = \binom{m+3}{4},$$

it seems reasonable to investigate whether, perhaps, in general,

$$a_{m+k} = (-1)^k \binom{m-1+k}{k}.$$

That this is, indeed, the case can easily be shown by induction. Assuming that the above formula for a_{m+i} is correct for $i=1, 2, \cdots, k-1$, one can write

$$a_{m+k} = 1 - \sum_{i=0}^{k-1} a_{m+i} \binom{m+k}{m+i}$$

or

$$a_{m+k} = 1 - \binom{m+k}{m} + \binom{m}{1}\binom{m+k}{m+1} - \binom{m+1}{2}\binom{m+k}{m+2} + \cdots + (-1)^k \binom{m+k-2}{k-1}\binom{m+k}{m+k-1}.$$

Using again identity (1), this expression can also be written as

$$a_{m+k} = (-1)^k \binom{m-1+k}{k}, \tag{6}$$

and since it has previously been shown that the formula holds for $i=1$, the induction is complete and the formula for the probability P_m can, therefore, be written as

$$P_m = \sum_{i=0}^{N-m} (-1)^i \binom{m-1+i}{i} S_{m+i}. \tag{7}$$

4. The derivation of the formula for $P_{[m]}$. If one puts

$$P_{[m]} = P_m - P_{m+1}$$

and substitutes the results obtained in the previous section, it follows that

$$P_{[m]} = \sum_{i=0}^{N-m} (-1)^i \binom{m-1+i}{i} S_{m+i} - \sum_{j=0}^{N-m-1} (-1)^j \binom{m+j}{j} S_{m+1+j}.$$

Substituting $i-1$ for j in the second summation

$$P_{[m]} = \sum_{i=0}^{N-m} (-1)^i \binom{m-1+i}{i} S_{m+i} - \sum_{i=1}^{N-m} (-1)^{i-1} \binom{m-1+i}{i-1} S_{m+i}$$

or

$$P_{[m]} = S_m - \sum_{i=1}^{N-m} (-1)^{i-1} S_{m+i} \left\{ \binom{m-1+i}{i} + \binom{m-1+i}{i-1} \right\}.$$

Using the well-known identity

$$\binom{m-1+i}{i}+\binom{m-1+i}{i-1}=\binom{m+i}{i},$$

this can also be written as

$$P_{[m]}=S_m+\sum_{i=1}^{N-m}(-1)^i\binom{m+i}{i}S_{m+i}$$

and the formula for $P_{[m]}$, the probability of the realization of exactly m among N events, can, finally, be written as

$$P_{[m]}=\sum_{i=0}^{N-m}(-1)^i\binom{m+i}{i}S_{m+i}. \tag{8}$$

A PROOF OF TAYLOR'S FORMULA

JAMES WOLFE, University of Utah

The following proof of Taylor's formula with remainder may seem more natural than the proofs ordinarily offered in calculus texts.

Suppose $f(x)$ has a continuous $(n-1)$st derivative in the closed interval between a and b and an nth derivative in the open interval between a and b. Then $f(x)$ can be approximated by a polynomial $p(x)$ of degree n which agrees in value with $f(x)$ at a and b and such that the first $n-1$ derivatives of p and f agree at a:

$$p(x)=f(a)+f'(a)(x-a)+\frac{f''(a)}{2!}(x-a)^2+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(x-a)^{n-1}+k(x-a)^n,$$

where k is determined so that $p(b)=f(b)$. Let $g(x)=f(x)-p(x)$. Then $g(x)$ and its first $n-1$ derivatives vanish at $x=a$ and also $g(b)=0$. Using successive applications of Rolle's theorem, $g'(x_1)=0$ for some x_1 between a and b, $g''(x_2)=0$ for some x_2 between a and x_1, *etc.* and finally $g^{(n)}(x_n)=0$ for some x_n between a and x_{n-1}. But $g^{(n)}(x)=f^{(n)}(x)-n!k$, consequently $k=f^{(n)}(x_n)/n!$ and

$$f(b)=p(b)=f(a)+f'(a)(b-a)+\cdots+\frac{f^{(n-1)}(a)}{(n-1)!}(b-a)^{n-1}+\frac{f^{(n)}(x_n)}{n!}(b-a)^n.$$

For $n=1$, this proof reduces to the usual proof of the mean value theorem.

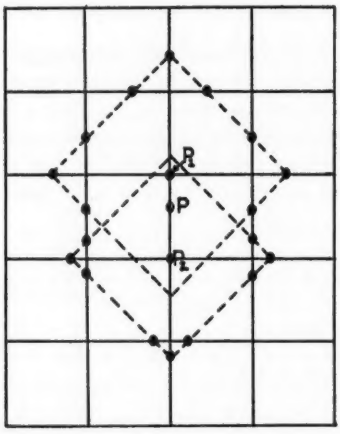
A NOTE ON THE TAXICAB GEOMETRY

H. J. CURTIS, Illinois Institute of Technology

In his guide book to the exhibition on geometry in the Museum of Science and Industry in Chicago, Menger* calls attention to an interesting non-euclidean plane geometry. He refers to it as a taxicab geometry. Its points are the points of a grid of streets which cuts a city into square blocks. The distance between two points, P_1 and P_2, is defined as the length of any shortest path *along streets* joining P_1 and P_2. In computing this distance we neglect the width of the streets. The circle $C(P, r)$ of radius r about a street corner P (*i.e.* the set of points which are at distance r from P by taxi) lies on a square in diamond position relative to the grid of the streets.

It is the purpose of this note to show that in this metric a circle, whose center may or may not be a corner, is not necessarily a part of the boundary of just one square, but that it may be part of the boundary of two squares.

We choose a Cartesian coordinate system the axes of which are streets. At least one coordinate of each point of the taxi geometry is an integer, if we take



as unit the length of a block. The street corners are the points with two integer coordinates. If $P_1=(x_1, y_1)$ and $P_2=(x_2, y_2)$, then the taxi distance between P_1 and P_2 is given, with two exceptions, by:

$$d(P_1, P_2) = |x_1 - x_2| + |y_1 - y_2|. \dagger$$

To describe the exceptional cases we let $z=[z]+z'$, where $[z]$ denotes, for every real number z, the largest integer $\leqq z$. The exceptions are:

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* Karl Menger, You Will Like Geometry, Chicago, 1952; p. 5.

† This is the case $n=1$ of Minkowski's distance

$$[|x_1 - x_2|^n + |y_1 - y_2|^n]^{1/n}.$$

1) If x_1 and x_2 are different integers and $[y_1]=[y_2]$, then $d(P_1, P_2)=|x_1-x_2|+\min(y_1'+y_2', 2-y_1'-y_2')$;
2) if y_1 and y_2 are different integers and $[x_1]=[x_2]$, then $d(P_1, P_2)=|y_1-y_2|+\min(x_1'+x_2', 2-x_1'-x_2')$.

We wish to describe $C(P, r)$ in case P is not a corner. Let P_1 and P_2 be the corners adjacent to P. Then, if $d(P, P_1)=k$, the set of points $C(P, r)$ is contained in the sum of the sets $C(P_1, r-k)$ and $C(P_2, r+k-1)$. For any corner Q, we denote the interior of the square $C(Q, s)$ as $D(Q, s)$. The circle of radius r about a point P, not a corner, is the intersection of the grid of streets with the octagon $D(P_1, r-k)+D(P_2, r+k-1)$. We see that $C(P, r)$ is not convex.

The accompanying figure illustrates the case $P=(0, 0.6)$, $P_1=(0, 1)$, $P_2=(0, 0)$ and $r=1.8$.

ELEMENTARY PROBLEMS AND SOLUTIONS

EDITED BY HOWARD EVES, State University of New York

Send all communications concerning Elementary Problems and Solutions to Howard Eves, Mathematics Department, Champlain College, Plattsburg, New York. This department welcomes problems believed to be new, and demanding no tools beyond those ordinarily furnished in the first two years of college mathematics. To facilitate their consideration, solutions should be submitted on separate, signed sheets, within three months after publication of problems.

PROBLEMS FOR SOLUTIONS

E 1071. *Proposed by Albert Wilansky, Lehigh University*

If $f(x)=\int_0^x \cos(1/t)dt$, show that $f'(0)=0$.

E 1072. *Proposed by S. J. Jasper, East Tennessee State College*

Find a four digit number in base ten which, when the order of the digits is reversed, becomes an equivalent number in base seven.

E 1073. *Proposed by G. W. Walker, Buffalo, N. Y.*

A polygonal spiral $A_1A_2A_3\cdots$ of unit segments winds counterclockwise and is constructed in the following manner. Point A_1 is at the origin, point A_2 is at $(1, 0)$, $\measuredangle A_{n-1}A_nA_{n+1}=2\pi/n$ for all $n\geqq 2$. Is there a point lying within the interior of each $\measuredangle A_{n-1}A_nA_{n+1}$? If, so, what are its coordinates?

E 1074. *Proposed by Vern Hoggatt, Oregon State College*

A regular n-gon with center O has a particle of mass m at each vertex. Let PO be a segment perpendicular to the plane of the n-gon and let l be a line

through P parallel to the plane of the n-gon. Show that the sum of the moments of inertia of the n masses about the line l is independent of the (restricted) orientation of l.

E 1075. *Proposed by H. S. Shapiro, Chatham, N. J.*

A sequence of N positive integers contains precisely n distinct numbers. If If $N \geqq 2^n$, show that it is possible to find a block of consecutive terms whose product is a square, and that this result need not hold if $N < 2^n$.

SOLUTIONS

Anagrams

E 1041 [1952, 696]. *Proposed by SHE SLID and O PARISH, Massachusetts Institute of Technology*

Surely the days of anagrams are not dead. Here are four well known mathematicians: (1) A DONUT SHIP, (2) SHE IS A NUT, (3) SEWER STAIRS, (4) FIRE ON SUB.

I. *U TOIL SON by GREW LIGHT SCAR, SOGGY OCEAN EELS LIT CELL.*[1] (1) THUDS PIANO was the father of the RAG BALE.[2] He reveled in indeterminate equations, so appropriately enough THUDS PIANO = A TOP HID SUN = I PUT SHAD ON = DOAN PUSH IT = PAN THIS DUO = A DONUT SHIP = DIOPHANTUS.

(2) SHE IS A NUT who produced THE LAST HOT MAMA'S PANICS.[3] Furthermore, USES A HINT = TIN HAS USE = SHUNS A TIE = THE USA SIN = SHE IS A NUT = STEINHAUS.

(3) WASTERS RISE and SLAY A SIN.[4] Hence we have WASTERS RISE = SERIES WARTS = SWEARS RITES = SEWER STAIRS = WEIERSTRASS.

(4) I BURN FOES = FIE ON BURS = I SOBER FUN = O FINE RUBS = FIRE ON SUB = FROBENIUS made an extensive study of CRUMBS NEAR A BILGE[5] where LAME-BRAIN CURS BEG,[5] BLUE MAN GRABS RICE,[5] and BARE GAL BURNS MICE,[5] A SLUMBERING BRACE.[5]

We also breathe new life into: (5) WENT ON, (6) LARCH STY, (7) ROTTED HUN, (8) REC'D ROE, (9) IV TENS, (10) KIND DEED, (11) THOU DREG, (12) CANE DOG, (13) SEE GUARDS, (14) PELT ONCE, (15) LIE CRUEL APE, (16) BAN ON RICH, (17) A FREE BUCH, (18) RICH CASE, (19) RAM NINE, (20) SING A TUNE, MOOR, (21) A POO IS NULL, (22) I REMOVED, (23) MAR CONE, (24) TY'S REVELS, (25) DEER GLEN, (26) CALL APE, (27) I RIVAL ACE, (28) HIM NOT AL, (29) SET CEDARS, (30) TIRE CHILD, (31) HARM A DAD, (32) I CHARM ALEC, (33) A CURT NO, (34) LEAN MUSE, (35) O NULL BIER, (36) REACH DIMES, (37) HI, SAY AH, (38) DRAB ROC, (39) ONE LIME, (40) CABIN COIF. Finally, we end AVEC CEVA, a pretty palindrome.

[1] *SOLUTION by CHARLES W. TRIGG, LOS ANGELES CITY COLLEGE,* [2] ALGEBRA,

[3] MATHEMATICAL SNAPSHOTS, [4] ANALYSIS, [5] ALGEBRAIC NUMBERS. (5) NEWTON, (6) CHRYSTAL, (7) TODHUNTER, (8) RECORDE, (9) STEVIN, (10) DEDEKIND, (11) OUGHTRED, (12) D'OCAGNE, (13) DESARGUES, (14) PONCELET, (15) PEAUCELLIER, (16) BRIANCHON, (17) FEUERBACH, (18) SACCHERI, (19) RIEMANN, (20) REGIOMONTANUS, (21) APOLLONIUS, (22) DE MOIVRE, (23) CREMONA, (24) SYLVESTER, (25) LEGENDRE, (26) LAPLACE, (27) CAVALIERI, (28) HAMILTON, (29) DESCARTES, (30) DIRICHLET, (31) HADAMARD, (32) CARMICHAEL, (33) COURANT, (34) MENELAUS, (35) BERNOULLI, (36) ARCHIMEDES, (37) HAYASHI, (38) BROCARD, (39) LEMOINE, (40) FIBONACCI

II. *Remarks by CHAS. PULL SINE,*[1] *Cooper Union.* As for the days of anagrams being dead, locate the names of 21 mathematicians of note in the following:

No gem in this or an epic either, yet as she said, "Here Tim, lend me my T-pole," Ess traced a regal nag. "Nice corpus," said lamed Bert, a critic, "you enthuze me—I am on ether, a piner for your recipe." "To be able to do a kind deed is bred in us," Ess argued, "but I lag so." "Ach, you tire, child. Put it down."

Dead indeed!

[1] PAUL L. CHESSIN, no gem = MONGE, or an epic = POINCARE, as she = HASSE, here Tim = HERMITE, lend me = MENDEL, my T-pole = PTOLEMY, Ess traced = DESCARTES, regal nag = LAGRANGE, nice corpus = COPERNICUS, lamed Bert = DALEMBERT, a critic = RICCATI, enthuze = ZEUTHEN, on ether = NOETHER, a piner = NAPIER, recipe = PEIRCE, able = ABEL, kind deed = DEDEKIND, bred in us = BURNSIDE, Ess argued = DESARGUES, I lag so = GALOIS, tire child = DIRICHLET.

III. *Addenda by H. W. Gould, Portsmouth, Va.* How about (1) YOUR SON IS OLD, (2) READ CHIMES, (3) AM SURE NOT GO IN, (4) HE TENSOR EAST, (5) CABIN FOCI, (6) ARYAN CASH A HARK, (7) RUN A CLAIM, (8) WHO IZ KALAMIR and (9) GO PAST HYAR?

(1) DIONYSODORUS, (2) ARCHIMEDES, (3) REGIOMONTANUS, (4) ERATOSTHENES, (5) FIBONACCI, (6) SHANKARACHARYA, (7) MACLAURIN, (8) AL-KHOWARIZMI, (9) PYTHAGORAS.

Also solved by Raphael Aronson, G. A. Baker, Jr., Leon Bankoff (BONE OF FLANK), T. A. Bickerstaff (STIFFER BACK), Julian Braun (A BURN), W. E. Briggs, J. E. Darraugh, William Douglas, D. M. Dribin, R. P. Eisinger (I IS GREEN), L. R. Ford (DR. OF), J. W. Forman, Louise Gartner, Bennington Gill (ENNOBLING GLINT), J. D. Haggard (HARD GAG), R. Huck, Meyer Jerison, J. H. Means, R. A. Miller (MLLE RI), George Millman, B. E. Mitchell (LITHE CLEM), Leo Moser, Hyman Orlin, A. F. Payton, L. L. Scott (ST. COT), William Small (MALLS), H. S. Wilf, R. H. Wilson, Jr., and the proposers (W. H. Shields and H. S. Shapiro). Late solutions by A. R. Hyde and J. V. Whittaker.

2-type Fibonacci Sequences

E 1042 [1952, 697]. *Proposed by P. L. Chessin, Cooper Union*

A sequence $\{a_n\}$, where a_0 and a_1 are given numbers, is defined recursively by $a_{n+2}=(a_{n+1}+a_n)/2$. Find $\lim_{n\to\infty} a_n$.

Solution by Leon Bankoff, Los Angeles, Calif. In the defined sequence, for all values of a_0 and a_1.

$$\lim_{n\to\infty} a_n = a_0 + (a_1 - a_0)(1 - 1/2 + 1/4 - 1/8 + \cdots)$$

$$= a_0 + (a_1 - a_0)(2/3) = (2a_1 + a_0)/3.$$

Also solved by P. M. Anselone, G. A. Baker, Jr., J. W. Baldwin, A. P. Boblétt, Julian Braun, L. Carlitz, J. E. Darraugh, Fred Discepoli, L. J. Dixon, H. H. Fox, H. L. Frisch, Lloyd Fulk, Harry Furstenberg, E. G. Goman and Vern Hoggatt (jointly), H. W. Gould, R. E. Greenwood, Arthur Gregory, Frank Herlihy, I. N. Herstein, Edwin Hewitt, H. W. Hickey, Verlin Hoberecht, M. H. Hoehn, R. Huck, A. R. Hyde, John Jones, Jr., M. L. Keedy, R. B. Kelman, M. S. Klamkin, A. E. Livingston, Lee Lorch, David Mandelbaum, Jerome Manheim, George Millman, Leo Moser, R. J. Oravec, Hyman Orlin, F. D. Parker, C. F. Pinzka, R. F. Reeves, L. A. Ringenberg, Saul Rosen, Joseph Rosenbaum, Arthur Rosenfeld, A. A. Sardinas, A. H. Smith, M. R. Spiegel, O. E. Stanaitis, D. D. Strebe, W. M. Stone, J. A. Tierney, C. W. Trigg, R. Z. Vause, Chih-yi Wang, R. E. Wild, H. S. Wilf, and the proposer. Late solutions by N. J. Fine, S. Parameswaren, and J. V. Whittaker.

A sequence $\{a_n\}$ defined recursively by $a_{n+2}=(a_{n+1}+a_n)/k$, where $k>0$, a_0, a_1 are given, may be called a *k-type Fibonacci sequence.* It can be shown that if $k<2$ then $a_n\to\pm\infty$, whereas if $k>2$ then $a_n\to 0$. It is interesting that the case of the problem, $k=2$, is exceptional.

Oravec pointed out a relation between the problem and a probability problem involving the tossing of a coin. Suppose a player tosses an ideal coin and is to score one point for every head turned up and two for every tail. He is to play until his score reaches or passes $n+2$. If p_n is his chance of attaining exactly n, then it can be verified that $p_{n+2}=(p_{n+1}+p_n)/2$.

Rosen considered the sequence $\{a_n\}$ defined by the recursion relation

$$a_n = (a_{n-1} + a_{n-2} + \cdots + a_{n-k})/k,$$

where k, a_0, a_1, $\cdots$, a_{k-1} are given.

Several solvers pointed out that the generalization in which $a_{n+2}=ra_{n+1}+sa_n$, where $r+s=1$, appears in Bromwich, *Theory of Infinite Series*, rev. 2nd ed., (1947), prob. 7, p. 22. Greenwood called attention to Gibson, *Advanced Calculus* (1931), prob. 5, p. 50, and Braun and Trigg called attention to prob. 2316, *School Science and Mathematics* [1952, 755]. Moser noted the interesting and somewhat novel method of solving linear difference equations given recently

by D. F. Lawden, "On the solution of linear difference equations," *Mathematical Gazette* [1952, 193–196].

Property of Three Concurrent Cevians

E 1043 [1952, 697]. *Proposed by O. J. Ramler, Catholic University, Washington, D. C.*

Prove that the sum of the ratios in which a point within a triangle divides the Cevians of this point is never less than 6 and that the product of the ratios is never less than 8.

Solution by C. W. Trigg, Los Angeles City College. The identity $x+1/x \equiv 2+(x-1)^2/x$ shows that the sum of a positive number and its reciprocal is always $\geqq 2$. Now if the Cevians divide the sides of the triangle in the ratios k, m, n then $kmn=1$ by Ceva's Theorem. Furthermore, the corresponding Cevians are divided by their common point in the ratios $n+1/m, k+1/n, m+1/k$ (see, *e.g.*, Altshiller-Court, *College Geometry*, 2nd ed. (1952), p. 163, Th. 342). It follows that the sum of these latter ratios,

$$(k + 1/k) + (m + 1/m) + (n + 1/n) \geqq 6.$$

Also, the product of these ratios,

$$(kmn + 1/kmn) + (k + 1/k) + (m + 1/m) + (n + 1/n) \geqq 8.$$

Since $kmn=1$ we observe that the product of the three ratios is always equal to the sum of the three ratios increased by 2. We also note that the minimum values of the sum and product of the three ratios will be attained if and only if the three Cevians are the three medians of the triangle.

Also solved by Leon Bankoff, Fred Discepoli, Harry Furstenberg, Louisa Grinstein, J. D. Haggard, M. S. Klamkin, B. Martin, George Millman, Martin Moliver, T. F. Mulcrone, E. J. Musch, Chih-yi Wang, and the proposer.

A Trigonometrical Product

E 1044 [1952, 697]. *Proposed by J. E. Wilkins, Jr., Nuclear Development Associates, Inc.*

Find

$$\prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^r.$$

Solution by J. A. Tierney, U. S. Naval Academy, Annapolis. Set

$$P = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^r.$$

Then we also have

$$P = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi(m-r)}{m}\right)^{m-r} = \prod_{r=1}^{m-1} \left(2 \sin \frac{\pi r}{m}\right)^{m-r},$$

whence

$$P^2 = \left(\prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m}\right)^m.$$

But it is known that

$$\prod_{r=1}^{m-1} 2 \sin \frac{\pi r}{m} = m. \tag{1}$$

Therefore $P = m^{m/2}$.

Also solved by M. J. Antchogno and Milton Handel (jointly), G. A. Baker, Jr., Julian Braun, C. N. Campopiano, L. Carlitz, Harry Furstenberg, H. W. Gould, M. H. Hoehn, Vern Hoggatt, P. G. Kirmser, M. S. Klamkin, A. E. Livingston, William Small, M. R. Spiegel, O. E. Stanaitis, Chih-yi Wang, R. E. Wild, and the proposer. Late solutions by N. J. Fine, S. Parameswaren, and J. V. Whittaker.

The following references were cited where formula (1) may be found: Bromwich, *Theory of Infinite Series*, p. 211; Durell and Robson, *Advanced Trigonometry*, p. 223; Franklin, *Treatise on Advanced Calculus*, prob. 42, p. 582; Nagell, *Introduction to Number Theory*, p. 173; Todhunter, *Plane Trigonometry*, p. 259; Whittaker and Watson, *Modern Analysis*, p. 240.

Gauss's Generalization of Wilson's Theorem

E 1045 [1952, 697]. *Proposed by S. W. Golomb, Harvard University*

Let $n!^{\phi}$ (pronounced "n-phi-torial") denote the product of all integers up to n which are prime to n. Then $n!^{\phi} \equiv \pm 1 \pmod{n}$. Prove this, and determine when the value is $+1$ and when it is -1.

Solution by William Small, University of Rochester. The result, known as Gauss's generalization of Wilson's theorem, is given, with proof, in Oystein Ore, *Number Theory and Its History*, theorem 11-6, p. 266. In the congruence one has the negative sign when $n = 4$, p^{α}, $2p^{\alpha}$, where p is an odd prime. In all other cases, one has the positive sign, except that for the case $n = 2$ it is immaterial which sign is used.

Also solved by J. W. Baldwin, P. T. Bateman, W. E. Briggs, L. Carlitz, L. C. Labowitz, J. Lehner, A. E. Livingston, Leo Moser, J. V. Whittaker, and the proposer.

Other references cited are Hardy and Wright, *Theory of Numbers*, theorem 129, p. 102, and Nagell, *Introduction to Number Theory*, theorem 79, p. 100.

ADVANCED PROBLEMS AND SOLUTIONS

EDITED BY E. P. STARKE, Rutgers University

Send all communications concerning Advanced Problems and Solutions to E. P. Starke, Rutgers University, New Brunswick, New Jersey. All manuscripts should be typewritten with double spacing and margins at least one inch wide. Problems containing results believed to be new or extensions of old results are especially sought. Proposers of problems should also enclose any solutions or information that will assist the editor. In general, problems in well known text books or results in readily accessible sources should not be proposed for this department.

PROBLEMS FOR SOLUTION

4528 [1953, 192] corrected. *Proposed by Paul Erdös, American University, Washington, D. C.*

Prove that the primes have the following property: If a sequence of consecutive integers n, $n+1$, $\cdots$, $n+k$ contains a prime p, then at least one of these integers is relatively prime to all others. Prove also that infinitely many integers which are not primes have the same property. (That sequences exist in which no member has this property has been shown by Sivasankaranarayana, Pillai and Szekeres.)

4543. *Proposed by Victor Thébault, Tennie, Sarthe, France*

Determine systems of numeration in which there are one or more four-digit squares of the form *aabb* and such that $a+b$ equals the sum of the digits of the square root.

4544. *Proposed by D. J. Newman, Harvard University*

The elements of a determinant are arbitrary integers. Determine the probability that the value of the determinant is odd.

4545. *Proposed by G. G. Lorentz, University of Toronto*

Let us say that a series, $\sum v_k$, $v_k \geqq 0$, is dominated by the series $\sum u_n$, $u_n \geqq 0$ if $v_k \leqq u_{n_k}$, $k=1, 2, \cdots$ for a sequence $n_1 < n_2 < \cdots$ of indices n. Construct a divergent series $\sum u_n$ such that each dominated series with decreasing terms is convergent.

4546. *Proposed by M. S. Klamkin, Polytechnic Institute of Brooklyn, New York*

Evaluate

$$I = \int_0^\infty \frac{\sin \frac{mx}{2} \sin \frac{nx}{2} \sin \frac{m+1}{2} x \sin \frac{n+1}{2} x}{x^2 \sin^2 \frac{x}{2}} \, dx$$

where m and n are integers and $m \geqq n$.

4547. *Proposed by H. S. Shapiro, Chatham, New Jersey*

Let $0 \leqq a_i \leqq 1$, $i=1, \cdots, n$ and put $A=\sum a_i$. Prove

$$\sum_{i=1}^{n} \frac{a_i}{1-a_i} \geqq \frac{nA}{n-A},$$

equality occurring only if all the a_i are equal.

SOLUTIONS

A Summation

4483 [1952, 254]. *Proposed by M. R. Spiegel, Rensselaer Polytechnic Institute, Troy, N. Y.*

Find the sum of

$$1+\frac{1}{2}\frac{1}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{1}{7^2}+\cdots.$$

Solution by Samuel Goldberg, Lehigh University. Consider the integral

$$\int_0^x \frac{\arcsin z}{z}\,dz$$

and expand the integrand in a power series in z. Then

$$\int_0^x \frac{\arcsin z}{z}\,dz = \int_0^x \left(1+\frac{1}{2}\frac{z^2}{3}+\frac{1\cdot 3}{2\cdot 4}\frac{z^4}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{z^6}{7}+\cdots\right)dz$$

$$= x+\frac{1}{2}\frac{x^3}{3^2}+\frac{1\cdot 3}{2\cdot 4}\frac{x^5}{5^2}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\frac{x^7}{7^2}+\cdots.$$

Inasmuch as the series in the integrand is convergent for $|z|<1$, the resulting power series in x is convergent for $|x|<1$. Moreover this series converges for $x=1$ and is then just the series whose sum, S, is to be found. By Abel's theorem,

$$S=\int_0^1 \frac{\arcsin z}{z}\,dz.$$

The substitution $z=\sin t$ yields the result in terms of a standard improper integral

$$S=-\int_0^{\pi/2} \log \sin t\,dt$$

which is easily evaluated by replacing $\sin t$ by $2 \sin \frac{1}{2}t \cos \frac{1}{2}t$. One obtains $S=\frac{1}{2}\pi \log 2$.

Also solved by T. M. Apostol, Ranko Bojanić, Leonard Carlitz, James Clunie, R. V. Esperti, Robert Frucht, Fritz Herzog, A. R. Hyde, H. Kaufman

and J. S. Shipman, J. B. Kelly, R. Kissling, M. S. Klamkin, A. M. Peiser and Sidney Katz, O. J. Ramler, Edward Saibel, O. E. Stanaitis, K. Subbarao and M. Perisastri, C. A. Swanson, F. Underwood, Eugene Usdin, Chih-yi Wang, J. E. Wilkins, Jr., and the Proposer.

Editorial Note. Bojanić and Carlitz point out that the present problem is included in question 330, *Journal of the Indian Mathematical Society*, 10, 1912, p. 32. On pp. 59–61 of the same issue, Ramanujan gives a general method for summing the series

$$1+\frac{1}{2}\frac{1}{3^n}+\frac{1\cdot 3}{2\cdot 4}\frac{1}{5^n}+\cdots.$$

Rational Approximations

4484 [1952, 254]. *Proposed by Murray Gerstenhaber and R. S. Varga, Harvard University*

Given a real number $y>0$. Let $g_n=h_n/n$ be the rational number which best approximates y, where h_n is a positive integer and $n=1, 2, \cdots$. Let

$$\lambda_n=\begin{cases}+ & \text{if } h_n/n-y\geqq 0,\\ - & \text{if } h_n/n-y<0.\end{cases}$$

Show that λ_n is periodic, *i.e.* there exists a positive integer k such that λ_{mk} is of one sign for all positive integers m, if and only if y is a positive rational number.

Solution by James Clunie, King's College, Aberdeen, Scotland. When y is rational there are positive integers p, q such that $y=p/q$, and therefore $h_n=np/q$ so that λ_n is periodic with period q. When y is irrational then for some arbitrary positive integer k suppose $\lambda_k=-$ ($\lambda_k=+$ is treated in an analogous manner). Hence $y=g_k+\alpha_k/k$ where $0<\alpha_k<\frac{1}{2}$. There exists an integer m such that $1>m\alpha_k>\frac{1}{2}$; ($m\alpha_k<\frac{1}{2}$, $(m+1)\alpha_k>1$ implies $\alpha_k>\frac{1}{2}$.) Therefore

$$y=g_k+\frac{m\alpha_k}{mk}=\frac{mkg_k+1}{mk}-\frac{(1-m\alpha_k)}{mk}=g_{mk}-\frac{(1-m\alpha_k)}{mk},$$

which shows that $\lambda_{mk}=+$. Thus no irrational number can give rise to a periodic λ_n.

Also solved by Henry Furstenberg, Joseph Lehner, Hugh Noland, Judy Richmond, L. M. Weiner, and the Proposers.

Generalization of the Droz-Farney Theorem

4485 [1952, 254]. *Proposed by Simon T. Kao, Catholic University of America, Washington, D. C.*

Through the orthocenter H of a triangle ABC draw any pair of perpendicular lines l_1 and l_2 and let A_1, A_2; B_1, B_2; C_1, C_2 be the respective points of intersection with the three sides BC, CA, AB. Show that the three points P, Q, R which

divide the three segments A_1A_2, B_1B_2, C_1C_2 in the same ratio r lie on a line.

I. *Solution by M. Perisastri, Maharajah's College, Vizianagram, South India.* Let $x/a_i+y/b_i=1$, $i=1, 2, 3$, be the equations of BC, CA, AB. Choose l_1 and l_2 as the x- and y-axes, so that H is $(0, 0)$. Then the coördinates of P, Q, and R are easily found to be

$$\left(\frac{a_i}{r+1}, \frac{rb_i}{r+1}\right).$$

Therefore the familiar determinant

$$|\Delta PQR| = \frac{r}{(1+r)^2}\begin{vmatrix} a_1 & b_1 & 1 \\ a_2 & b_2 & 1 \\ a_3 & b_3 & 1 \end{vmatrix}.$$

Using the condition that H is the orthocenter we get

$$\begin{aligned} a_1a_2a_3(b_2-b_3)+b_1b_2b_3(a_2-a_3) &= 0, \\ a_1a_2a_3(b_3-b_1)+b_1b_2b_3(a_3-a_1) &= 0, \\ a_1a_2a_3(b_1-b_2)+b_1b_2b_3(a_1-a_2) &= 0. \end{aligned}$$

Dividing these equations respectively by a_2a_3, a_3a_1, a_1a_2 and adding, we have $a_1b_2-a_1b_3+a_2b_3-a_2b_1+a_3b_1-a_3b_2=0$, whence $|\Delta PQR|=0$ and the points P, Q, R are collinear.

II. *Solution by O. J. Ramler, Catholic University of America, Washington, D. C.* Consider the parabola determined by the sides of triangle ABC and a line l_1 through its orthocenter H. Then by an elementary property of the parabola the line l_2 perpendicular to l_1 at H will also be tangent to the parabola. Tangents to the parabola intersect the sides of the triangle in projectively related ranges of points. These projectivities may be considered to be determined by $(A_1A_2I_a)$, $(B_1B_2I_b)$ and $(C_1C_2I_c)$ on sides BC, CA, AB respectively, where I_a, I_b, I_c are the ideal points on those sides. If P_a, P_b, P_c divide segments A_1A_2, B_1B_2, C_1C_2 in the same ratio, then the cross ratios $(A_1A_2P_aI_a)$, $(B_1B_2P_bI_b)$, and $(C_1C_2P_cI_c)$ are equal, thus identifying P_a, P_b, P_c as corresponding points in the three projectively related ranges on the sides of the triangle. Hence P_a, P_b, P_c are collinear, and the envelope of this line is the parabola for varying ratios. Its focus is the Miquel point of the quadrilateral consisting of the sides of the triangle and line l_1.

Also solved by Joseph Langr, Sedalia M. Sims, Roscoe Woods, and the Proposer.

Editorial Note. Woods demonstrates that the orthocenter is the only point which possesses the cited property.

Arbitrarily Large Entire Functions

4486 [1952, 254]. *Proposed by D. J. Newman, Harvard University*

Let $f(x)$ be any function continuous on the positive axis. Prove that there exists an entire function which is always larger than this $f(x)$ along the positive axis. (That is, prove that there are arbitrarily large entire functions.)

Solution by Joshua Barlaz, Rutgers University. The result is classic, occurring as a lemma in a paper by Poincaré, Sur les fonctions à espaces lacunaires, *American Journal of Mathematics*, xiv, 1892, pp. 213–215. A proof by Borel (*Leçons sur les séries à termes positifs*, Paris, 1902, p. 27) is quoted by Hardy, *Orders of Infinity*, Cambridge, 1924, pp. 10–11. Implicit in Poincaré's work is the following generalization: *Let $F(x)$ be continuous for all real x, then there exists an integral function $f(z)$ such that $F(x) \geqq f(x)$ for all real x.* Borel's proof is easily adapted to establish this result.

Define $g(x) = \max\{f(x), f(-x)\}$; $h(x) = \max g(t)$ for $0 \leqq t \leqq x$. $g(x)$ is an even function, $g(x)$ and $h(x)$ are continuous for $x \geqq 0$, $h(x) \geqq g(x)$, $h(x)$ is monotone, and $h(x) \uparrow \infty$ as $x \to \infty$. Let $\phi(x) = h^2(x)$. Then $\phi(x) \geqq h(x)$ for $x \geqq x_0$. Choose two sequences of positive numbers $\{a_n\}$, $\{b_n\}$, such that

$$b_1 < a_1 < b_2 < a_2 < b_3 < a_3 < \cdots$$

and such that a_n, $b_n \to \infty$. We can now choose a sequence of positive even integers ν_n so that both

$$\nu_{n+1} > \nu_n \quad \text{and} \quad \left(\frac{a_n}{b_n}\right)^{\nu_n} > \phi(a_{n+1}).$$

Next, let $F^*(x) = \sum_{n=1}^{\infty} (x/b_n)^{\nu_n}$. The series in convergent for all x. Moreover, if $a_n \leqq x < a_{n+1}$,

$$F^*(x) > \left(\frac{a_n}{b_n}\right)^{\nu_n} > \phi(a_{n+1}) \geqq \phi(x) \geqq h(x)$$

for $x \geqq \max(x_0, 1)$. Let

$$M = \max_{0 \leqq x \leqq \max(x_0, 1)} h(x), \qquad F(x) = F^*(x) + M.$$

Then $F(x) \geqq h(x) \geqq g(x)$ for all $x \geqq 0$. From the fact that $F(x)$ and $g(x)$ are even functions it now follows that $F(x) \geqq f(x)$ for all real x.

Also solved by Ranko Bojanić, Melvin Henriksen, Fritz Herzog, Jacob Korevaar, Jack Kotik and the proposer, George Piranian, W. R. Scott, and E. M. Wright.

Editorial Note. Piranian points out that the original hypothesis should read: let $f(x)$ be any function continuous for $x \geqq 0$.

Divergent Series, Partial Sums

4487 [1952, 254]. *Proposed by Paul Erdös, American University, Washington, D. C.*

Let

$$\sum_{k=1}^{\infty} d_k = \infty, \qquad d_k > 0, \qquad D_n = \sum_{k=1}^{n} d_k;$$

$$\sum_{k=1}^{\infty} a_k < \infty, \qquad a_k > 0.$$

Further assume that $a_k/d_k \geqq a_{k+1}/d_{k+1}$. Prove

$$\lim_{k\to\infty} \frac{a_k D_k}{d_k} = 0.$$

Solution by Ranko Bojanić, Mathematical Institute, Belgrade, Yugoslavia. We prove a somewhat more general result obtained by replacing the hypothesis on $a_k/d_k - a_{k+1}/d_{k+1}$ by the following:

$$\frac{a_k}{d_k} - \frac{a_{k+1}}{d_{k+1}} \geqq -M\frac{d_k}{D_k^2}, \tag{1}$$

where M is a constant $\geqq 0$. Take $\lambda > 1$. Take m such that

$$D_{m-1} \leqq \lambda D_{n-1} < D_m. \tag{2}$$

From the identity

$$\frac{a_m D_m}{d_m}\sum_{k=n}^{m}\frac{d_k}{D_k} = \sum_{\nu=n}^{m}\left(\frac{a_m D_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu}\right)\frac{d_\nu}{D_\nu} + \sum_{k=n}^{m} a_k,$$

it follows that

$$\frac{a_m D_m}{d_m}\sum_{k=n}^{m}\frac{d_k}{D_k} \leqq \max_{n\leqq\nu\leqq m}\left(\frac{a_m D_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu}\right)\sum_{k=n}^{m}\frac{d_k}{D_k} + \sum_{k=n}^{m} a_k.$$

Now the inequality (1) may be written as follows:

$$\frac{a_{k+1}D_{k+1}}{d_{k+1}} - \frac{a_k D_k}{d_k} \leqq M\frac{d_k}{D_k} + a_{k+1}.$$

Since

$$\frac{d_k}{D_k} = 1 - \frac{D_{k-1}}{D_k} \leqq \log\frac{D_k}{D_{k-1}},$$

which follows obviously from $1-x \leqq \log(1/x)$ for $x \leqq 1$, we obtain

$$\frac{a_{k+1}D_{k+1}}{d_{k+1}} - \frac{a_kD_k}{d_k} \leqq M \log \frac{D_k}{D_{k-1}} + a_{k+1}.$$

Summing this inequality from ν to $m-1$ we obtain

$$\frac{a_mD_m}{d_m} - \frac{a_\nu D_\nu}{d_\nu} \leqq M \log \frac{D_{m-1}}{D_{\nu-1}} + \sum_{k=\nu}^{m} a_k.$$

Thus

$$\begin{aligned}\frac{a_mD_m}{d_m} \sum_{k=n}^{m} \frac{d_k}{D_k} &\leqq \max_{n \leqq \nu \leqq m} \left(M \log \frac{D_{m-1}}{D_{\nu-1}} + \sum_{k=\nu}^{m} a_k\right) \sum_{k=n}^{m} \frac{d_k}{D_k} + \sum_{k=n}^{m} a_k \\ &\leqq \left(M \log \frac{D_{m-1}}{D_{n-1}} + \sum_{k=n}^{m} a_k\right) \sum_{k=n}^{m} \frac{d_k}{D_k} + \sum_{k=n}^{m} a_k,\end{aligned}$$

or

$$\frac{a_mD_m}{d_m} \leqq M \log \frac{D_{m-1}}{D_{n-1}} + \sum_{k=n}^{m} a_k + \left(\sum_{k=n}^{m} \frac{d_k}{D_k}\right)^{-1} \sum_{k=n}^{m} a_k.$$

Further, by (2) we have

$$\sum_{k=n}^{m} \frac{d_k}{D_k} \geqq \frac{1}{D_m} \sum_{k=n}^{m} d_k = 1 - \frac{D_{n-1}}{D_m} > 1 - \frac{1}{\lambda} > 0,$$

and $D_{m-1}/D_{n-1} \leqq \lambda$. Hence

$$\frac{a_mD_m}{d_m} \leqq M \log \lambda + \sum_{k=n}^{m} a_k + \frac{\lambda}{\lambda - 1} \sum_{k=n}^{m} a_k.$$

Here let $n \to \infty$. Since $m \to \infty$ with n and $\sum_1^\infty a_k$ converges, we obtain finally

$$\limsup_{m \to \infty} \frac{a_mD_m}{d_m} \leqq M \log \lambda.$$

But λ may be chosen arbitrarily near to 1, and the required result follows.

Also solved by James Clunie, Henry Furstenberg, Cecil Hastings, Jr., Stanley Katz and A. M. Peiser, Jacob Korevaar, A. E. Livingston, C. J. Rajagopal, A. J. Rodriguez, P. Somanadham and M. Perisastri, O. E. Stanaitis, Otto Szász, Gabor Szegö, Chih-yi Wang, R. H. Breusch, and E. M. Wright.

Editorial Note. Results which generalize the above proposal in slightly different ways have been given by C. T. Rajagopol, *Mathematics Student*, vol. 8, pp. 118–123, and by S. Minakshisundaram, *Mathematics Student*, vol. 9, pp. 78–81.

RECENT PUBLICATIONS

EDITED BY E. P. VANCE, Oberlin College

All books for review should be sent directly to the editor of this department, American Mathematical Monthly, Oberlin College, Oberlin, Ohio, and not to any of the other editors or officers of The Association.

Application of Linear Programming to the Theory of the Firm. By R. Dorfman. California, University of California Press, 1951. ix+98 pp. $3.50.

The linear programming problem is maximization of a linear form cx subject to linear inequalities $Ax \leqq b$, $x \geqq 0$. Dually, one minimizes wb subject to $wA \leqq c$, $w \geqq 0$; either problem has a finite solution if and only if the other does and the optimal values of cx and wb are equal (Tucker, Kuhn and Gale). Here x is a vector of "activity" level; c is a net profit vector, b a restrictions vector (*e.g.*, on capacities), w a vector of imputed valuations.

The activities or "processes" of which the firm is composed are *pre-selected* as in the activity analysis of Koopmans (*cf. Activity Analysis of Production and Allocation*, T. Koopmans editor, J. Wiley and Sons, 1951). Indeed, aside from Chapter III on monopoly and monopsony (non-linear in having quadratic rather than linear forms), Dorfman's specialization to the firm neither introduces new concepts nor obtains new results associated with the specialization. Even in Chapter III, it might be added, Dorfman ignores or overlooks the effective alternative of treating his problems, which involve change in unit costs (or prices) at various levels, in a linear programming formulation, through introduction of new variables and inequalities. Further, the argument (p. 57) for an optimal number of processes employed by a monopolist is deficient since it assumes ("as we have seen previously") linearity of the form maximized.

Nowhere is there to be found intuitive geometric formulation of the problems; nowhere (although the cover speaks glibly of practical applications) is there developed a single realistic application. Geometrically, the solutions to such problems form a bounded, closed polyhedral convex set; a linear form takes on its maximum at an extreme point of such a set, a quadratic form may do so at an internal point—these highly relevant facts to the highly emphasized "number of processes" theorems do not emerge in the matricial obfuscation of Chapters II and III. Nor does a solution of the general computational problem (the "outstanding problem" in 1951)—solved by the reviewer (*Econometrica*, April 1952) with simple geometry—appear.

In chapter IV on "assumptions, limitations and possibilities," the author confounds (p. 83, p. 88) an infinite range of alternatives with the impossiblity of description by a finite set of inequalities, thus apparently ruling out treatment of "oil refining and the chemical industries," which have since been successfully treated. His basic conception of the divisibility assumption (non-integer solutions admissible) as "not seriously restrictive" in terms of a "few indivisible items" such as ships overlooks completely the necessity for Diophantine (lattice-point) optima in many machine scheduling problems, or the automatic

generation of such in the important transportation problem.

In brief, the chief deficiency of the book is its easy superficiality which on the one hand narrows to a seriously restrictive misconception of practical applicability of linear programming (*e.g.*, basic assumption 1, p. 18) and on the other bursts all bounds in identification of the imputed valuations of the dual problem with cost accounting quantities.

Despite its limitations the book is a step in the right direction. There is little doubt that linear programming may be used to extend and enrich the economic theory (and practice) of the business firm.

A. CHARNES
Carnegie Institute of Technology

Statistical Theory with Engineering Applications. By A. Hald. John Wiley and Sons, Inc., New York, 1952. xii+783 pages.

The purpose of the book is to provide engineers with the statistical techniques most important in their work and the theory underlying these techniques. A knowledge of calculus is presupposed.

Before introducing applied techniques, the book presents an introduction to mathematical statistics: elementary probability theory, fundamental properties of distributions, special distributions, limit theorems and sampling distributions. Subsequent chapters are devoted to the applications of special sampling distributions such as the binomial, Poisson, normal, Chi-Square, "*t*," variance ratio and range. Regression and correlation theory are very thoroughly treated and the standard techniques in the analysis of variance are covered. A chapter is also devoted to sampling from finite populations.

The book is large and remarkably comprehensive; it is chock-full of techniques which are useful in the analysis of data. The chapters on regression, for example, discuss regression when the variance is proportional to a function of the independent variate and give a very interesting discussion of choice between different regression curves. The book contains material on growth curves, fitting truncated and mixed normal distributions, on errors of measurement, on skew distributions and normalizing transformations, on tolerance limits and many other topics of importance in engineering research not ordinarily covered.

The book is unique in combining a comprehensive collection of applied techniques and theory. The customary procedure is simply to list the techniques and, at most, justify them intuitively. On the other hand, many people feel that statistics is a mathematical subject and can be properly understood only if at least some of the basic derivations are studied. It is surprising that more of the statistics books for engineers do not draw on the mathematical background engineers are required to take, and Hald's book appears to be a unique and extremely valuable addition to our reservoir of statistics texts.

A. H. BOWKER
Stanford University

Linear algebra and matrix theory. By. R. R. Stoll. McGraw-Hill Book Company, New York, 1952. 16+272 pp., $6.00.

This is an excellent introduction to the concepts and techniques of modern algebra, via the usual material on linear equations and reduction of matrices to canonical form. For example, in the course of the development the author pauses to define and discuss the abstract concepts of field, vector space, equivalence relation, group, and ring when these concepts arise naturally. The book is written with care and rigor, coupled with enough numerical examples, exercises, and passing from the special to the general to prevent the abstractions from floating out of reach. This nice balance is exemplified by the fact that, while matrices and n-tuples occupy most of the space in the book, one is left with the feeling that linear mappings and abstract vector spaces have been the real objects of study.

The topics covered are: Chapter 1: systems of linear equations; Chapter 2: abstract finite dimensional vector spaces, their dimensions, subspaces, direct sums and quotient spaces (undergraduates unskilled in abstract thinking may find this a little difficult); Chapter 3: matrices, rank, row- and column-equivalence, multiplication; Chapter 4: determinants, introduced axiomatically, following Schreier; Chapter 5: bilinear, quadratic, and Hermitian functions and forms; Chapter 6: generalities about groups and rings of linear transformations; Chapter 7: reduction to canonical form under similarity; Chapter 8: vector spaces with inner products, the principal axis theorem with the noteworthy feature that Hermitian and symmetric transformations are treated as special cases of normal transformations.

The approach in general is fresh but not radical, and should be stimulating to most teachers as well as to students who can handle some degree of abstraction. The over-all tone is one of satisfying completeness.

DANIEL ZELINSKY
Northwestern University

Statistical Theory in Research. By R. L. Anderson and T. A. Bancroft. McGraw-Hill Book Company, New York, 1952, xvi+399 pages. $7.00.

This book is intended as a text and reference book in statistical methods for research workers. A knowledge of calculus is assumed. Useful suggestions are given on the selection of material for a one year course. A well chosen set of problems is given at the end of each chapter. The standard methods of estimation and hypothesis testing are presented in such a way as to give the student a good working knowledge of the subject.

Since the book is not intended for students specializing in statistics, the authors have not tried to give a rigorous presentation, nor a critical discussion of the basic concepts involved. By leaving out the technical details, they are able to give a coherent and very readable account.

One point on which questions may be raised is the omission of recent work

in statistics. Concepts introduced by Wald, such as risk function, minimax estimation and sequential analysis, do not appear. It is the reviewer's opinion that these ideas and some of their simplest applications have an important place in even the most elementary statistical methods book. Less important is the absence of any mention of the modern logical emphasis on statistical decision making as opposed to statistical inference. However, on page 6 where the term "inductive inference" is used it might be well to refer the reader to the introduction of Neyman's *First Course in Probability and Statistics*.

Of minor suggestions and minor errors noted, the following may be mentioned: Page 31—The definitions of $E(x)$ and $E[\theta(x)]$ are contradictory and are given for bounded random variables only, while examples such as 4.21 involve the unbounded case. Additional explanation would be helpful. Page 41—Conditions analogous to the univariate case for $F(x, y)$ to be a cumulative distribution do not exist, as Gnedenko's counter-example shows. Page 94—A consistent estimator need not be unbiased in the limit. Page 101—Additional explanation would prepare the reader for cases, such as the rectangular, in which the maximum likelihood estimate is not a solution of $\partial L/\partial\theta = 0$, and for solutions of $\partial L/\partial\theta = 0$ which do not give a maximum.

The authors are to be congratulated on having written an excellent book for the purposes intended. It will be particularly useful as a text for courses in applied statistics.

C. R. BLYTH
University of Illinois

NEW BOOKS RECEIVED

Student's Workbook for Essential Business Mathematics. Second Edition. By Llewllyn R. Snyder. New York, McGraw-Hill Book Company, 1953. vi+153 pages. $2.50.

An Introduction to Mathematical Thought. By E. R. Stabler. Cambridge, Mass., Addison-Wesley Publishing Company, 1953. 20+268 pages. $4.50.

Elements of Mathematics. By Helen M. Roberts and Doris S. Stockton. Cambridge, Mass., Addison-Wesley Publishing Company, 1953. 8+211 pages. $3.00.

Philosophy and Psycho-Analysis. By John Wisdom. New York, Philosophical Library, 1953. vi+282 pages. $5.75.

Construction and Applications of Conformal Maps. Edited by E. F. Beckenbach. (Proceedings of a Symposium held at the Institute for Numerical Analysis of the National Bureau of Standards at the University of California, June 22–25, 1949.)

Logic for Mathematicians. By J. B. Rosser. New York, McGraw-Hill Book Company, 1953. xiv+530 pages. $10.00.

Demand Analysis, A Study in Econometrics. By Herman Wold in association with Lars Jureen. New York, John Wiley and Sons, Inc., 1953. xvi+358 pages, $7.00.

Complex Analysis. By L. V. Ahlfors. New York, McGraw-Hill Book Company, 1953. xi+247 pages. $5.00.

Solid Geometry. By W. G. Shute, W. W. Shirk, and G. F. Porter. New York, American Book Company, 1953. vii+280 pages. $2.48.

An Introduction to Linear Programming. By A. Charnes, W. W. Cooper, and A. Henderson. New York, John Wiley and Sons, Inc. 1953. ix+74 pages. $2.50.

Numerical Solution of Differential Equations. By W. E. Milne. New York, John Wiley and Sons, Inc., 1953. xi+275 pages. $6.50.

CLUBS AND ALLIED ACTIVITIES

EDITED BY H. D. LARSEN, Albion College

Send reports of club projects, bibliographies of program topics, expository articles, curiosa, descriptions of career opportunities, and other material of interest to clubs and undergraduate students to H. D. Larsen, Albion College, Albion, Michigan.

CLUB NEWS

(Clubs are invited to contribute news items)

The Mathematics Society of the Massachusetts Institute of Technology held a competition for Freshmen on December 6, 1952. Approximately 100 students completed a three hour examination which emphasized concepts rather than technique. The three prize winners were: first, Max A. Plager; second, William G. Strang; third, Donald A. Bavly.

The Kansas Gamma chapter of Kappa Mu Epsilon, Mount St. Scholastica College, announces a new type of prize contest which aims to acquaint students with mathematics journals. An award will be given to the student who demonstrates superior achievement in an examination over the material of a non-technical nature contained in *The Pentagon*, *Mathematics Magazine*, *School Science and Mathematics*, and *The Mathematics Teacher*.

TESTING FOR PRIMALITY

LEONARD CANERS, St. Michael's College,

To determine whether a number C is prime or not, we may use a table of factors and primes.* Such a table, however, is limited to values of C which do not exceed the last entry. The following device enables us to double the scope of the table in a relatively simple manner.

The method depends on the rather elementary fact that if $A+B=C$ and $A-B=R$ (all letters representing positive integers), then if R is a divisor of B it is a divisor of A and therefore of C. On the other hand, if R is not a divisor of B it is not a divisor of A and therefore not of C.

* For instance, see Mathematical Tables from Handbook of Chemistry and Physics.

As an example of the procedure, let our table of factors and primes include numbers from 2 to 2009, and let it be required to test $C=2491$ for primality. As usual, we only need test as divisors the prime numbers not greater than $\sqrt{2491}=49+$. Accordingly, we set in a column the odd primes up to 49; these are the values of R to be tested. Now if $A-B=R$ and $A+B=C$, then $B=\frac{1}{2}(C-R)$, from which the initial value of B can be computed: when $R=3$, $B=\frac{1}{2}(2491-3)=1244$. To complete the B-column, we write in succession 1244, 1243, 1242, $\cdots$, skipping where an odd number is missing in the R-column.

R	B	Prime divisors of B	R	B	Prime divisors of B
3	1244	2, 311	.	.	.
5	1243	11, 113	29	1231	prime
7	1242	2, 3, 23	31	1230	2, 3, 5, 41
.	.	.	.	.	.
11	1240	2, 5, 31	.	.	.
13	1239	3, 7, 59	37	1227	3, 409
.	.	.	.	.	.
17	1237	prime	41	1225	5, 7
19	1236	2, 3, 103	43	1224	2, 3, 17
.	.	.	.	.	.
23	1234	2, 617	47	1222	2, 13, 47
.	.	.			

The prime divisors of B are now read from our table of factors and primes. (In practice, these divisors are merely noted in the table. They are written down here to clarify the method.) If B does not have the prime divisor R opposite it, that value of R can be struck out. If any value of R is not struck out, that number is a divisor of C. If all the values of R are struck out, C is prime. In the example we note that 1222 has the divisor 47, whence 2491 has the divisor 47 and is not prime.

NEWS AND NOTICES

EDITED BY EDITH R. SCHNECKENBURGER, University of Buffalo

Readers are invited to contribute to the general interest of this department by sending news items to Edith R. Schneckenburger, University of Buffalo, Buffalo 14, New York. Items must be submitted at least two months before publication can take place.

CONFERENCE ON FUNCTIONS OF A COMPLEX VARIABLE

The Mathematics Department of the University of Michigan is sponsoring a conference on functions of a complex variable with emphasis on topological

methods. The Conference will be held June 17–30, 1953 in Ann Arbor, Michigan.

The program will consist of a group of invited addresses and of sessions for twenty-minute contributed papers. Among those who will present hour addresses are the following: L. V. Ahlfors, L. Bers, A. Beurling, S. Bochner, S. Kakutani, C. Loewner, M. Morse, Z. Nehari, R. Nevanlinna, M. Ohtsuka, P. Rosenbloom, H. L. Royden, L. Sario, D. C. Spencer, G. T. Whyburn.

The Summer Session of the University of Michigan (June 22–August 15) will devote special attention to complex variable theory. In particular, Professor Rolf Nevanlinna will conduct a course on advanced theory of functions of a complex variable and two seminars in this field.

For further information write to Professor Wilfred Kaplan, 274 West Engineering Building, University of Michigan, Ann Arbor, Michigan.

CONFERENCE FOR TEACHERS OF MATHEMATICS

The third annual Conference for Teachers of Mathematics will be held on the Los Angeles Campus of the University of California from July 6–17, 1953. The popular mathematics laboratory course given under the leadership of Miss Ida M. Bernhard will be offered again in conjunction with the Conference. Further information and application blanks may be obtained by writing to Professor Clifford Bell, Mathematics Department, University of California, Los Angeles 24, California.

INSTITUTE FOR MATHEMATICS TEACHERS

An Institute for Mathematics Teachers will be held at the University of Michigan, Ann Arbor, Michigan on August 3–14, 1953. This Institute is sponsored by the School of Education and the Department of Mathematics of the University with the cooperation of the Michigan Council of Teachers of Mathematics. The program includes lectures by representatives of various industries, field trips, discussion and laboratory groups. For further information write to Professor P. S. Jones, Angell Hall, University of Michigan, Ann Arbor, Michigan.

SUMMER SESSION AT REED COLLEGE

Under sponsorship of the Fund for the Advancement of Education, Reed College and the Portland (Oregon) Public Schools are cooperating to enrich the school program for gifted children. A special seminar in mathematics and physical science will be offered this summer at Reed College, primarily for teachers in the program, but other interested teachers will be welcome. Directors of the seminar are Professor Emeritus E. T. Bell of California Institute of Technology, Professors K. E. Davis and R. A. Rosenbaum of Reed College. For further information write to the Director of the Summer Session, Reed College, Portland, Oregon.

PERSONAL ITEMS

Associate Professor Arthur Bernhart of the University of Oklahoma has received a five hundred dollar award "for extraordinary excellence in student counseling and in the teaching of freshman and sophomore students." A grant of $25,000 to the University will make it possible to present ten similar awards each year during the next five years.

Professor O. E. Neugebauer of Brown University has been awarded the first Dannie Heineman Prize for his book *The Exact Sciences in Antiquity* by the Heineman Foundation for Research, Educational, Charitable and Scientific Purposes. This Foundation will continue to present a prize of $5,000 every three years to the author of an outstanding book on a high scientific level in the mathematical and physical sciences.

Associate Professor Marguerite Lehr of Bryn Mawr College and Professor I. J. Schoenberg of the University of Pennsylvania represented the Association at the meeting of the American Academy of Political and Social Sciences in Philadelphia, Pennsylvania on April 10–11, 1953.

Professor S. Grace Smyth, Associate Professor of Mathematics and Dean of Women at Knox College, served as the representative of the Association at the inauguration of President R. W. Gibson of Monmouth College on April 17, 1953.

Dr. E. L. Arnoff, formerly Aeronautical Research Scientist with the National Advisory Committee for Aeronautics, Cleveland, Ohio, is now a research associate with the Operations Research Group, Case Institute of Technology.

Mr. R. S. Barton has accepted a position as a senior mathematician with International Business Machines Corporation, Poughkeepsie, New York.

Dr. R. L. Beinert of Hobart and William Smith Colleges has been promoted to an assistant professorship.

Mr. James Bercos, who was formerly with the United States Naval Ordnance Plant, Macon, Georgia, is now a personnel statistician with Lockheed Aircraft Corporation, Marietta, Georgia.

Professor L. M. Blumenthal has returned to his position at the University of Missouri after serving as a consultant at the Institute for Numerical Analysis.

Mr. O. C. Braune, previously a student at St. Mary's University, has accepted a position as an electrical engineer with the Sandia Corporation, Albuquerque, New Mexico.

Mr. J. E. Brown of Florida State University has been appointed to an instructorship at the University of Georgia.

Dr. W. B. Brown, formerly with the National Advisory Committee for Aeronautics, Cleveland, Ohio, is now a design specialist with Northrop Aircraft, Hawthorne, California.

Assistant Professor Emalou Brumfield of Kent State University has been on leave of absence from the University during 1952–53 and has been doing graduate work at Ohio State University.

Mr. G. R. Bushyeager has been appointed Professor and Head of the Department of Mathematics of Morningside College.

Dr. R. E. Carr, who was formerly a research engineer with North American Aviation, Downey, California, has been appointed Head of the Computer Operations Group, Jet Propulsion Laboratory, California Institute of Technology.

Mr. J. L. Connors, previously a graduate student at Rutgers University, has a position as a mathematician with Project Meteor, Massachusetts Institute of Technology.

Mr. G. R. Costello has accepted a position as Aerophysics Project Engineer with Chance-Vought Aircraft, Dallas, Texas.

Mr. J. I. Derr of Southern Methodist University has a position as Aerophysics Engineer with Consolidated Vultee Aircraft Corporation, Fort Worth, Texas.

Mr. G. M. Dillon is a mathematics specialist in the Treasury Division of E. I. du Pont de Nemours and Company, Wilmington, Delaware.

Mr. Raymond Doby, formerly a development engineer with Sylvania Electric Products, Kew Gardens, New York, has accepted a position as a research engineer with Burroughs Adding Machine Company, Philadelphia, Pennsylvania.

Dr. I. A. Dodes, who has been teaching at Stuyvesant High School, New York City, has been appointed Chairman of the Department of Mathematics of Morris High School, Bronx, New York.

Professor R. D. Doner of Alabama Polytechnic Institute is now Chief Training Officer at Redstone Arsenal, Huntsville, Alabama.

Graduate Assistant Edwin Duda of West Virginia University has been promoted to an instructorship.

Mr. R. P. Eddy, previously of the Naval Ordnance Laboratory, White Oak, Silver Spring, Maryland, has a position as a mathematician at the David Taylor Model Basin, Washington, D. C.

Mr. Harvey Eisenberg, who was located previously at Fort Monmouth, New Jersey, has accepted a position as a mathematician with Evans Signal Laboratory, Belmar, New Jersey.

Mr. G. V. Emerson, formerly a student at Hobart College, is engaged now as an applied mathematician with Atlantic Research Corporation, Alexandria, Virginia.

Assistant Professor Virginia I. Felder of Mississippi Southern College is teaching in the United States Armed Forces Institute, Japan.

Mr. E. A. Franz, who has been on active military duty in the Signal Corps, United States Army, has returned to Culver-Stockton College as Assistant Professor of Mathematics.

Assistant Professor I. C. Gentry of Wake Forest College has been promoted to an associate professorship.

Mr. R. D. Glauz, previously a research assistant at Brown University, is now a staff member at the Los Alamos Scientific Laboratory.

Mr. W. A. Golomski has been appointed to an instructorship at St. Louis University.

Dr. R. E. Graves, who was formerly at White Sands Proving Ground, Las Cruces, New Mexico, has accepted a position as a mathematician with Goodyear Aircraft Corporation, Litchfield, Arizona.

Mr. E. LeR. Grindall has a position as a project engineer at the Ordnance Research Laboratory, Pennsylvania State College.

Mr. J. P. Gwin, previously a student at the University of Tulsa, is now a computer with the Seismograph Service Corporation, Tulsa, Oklahoma.

Mr. W. C. Hamilton has received an appointment at the Crellin Laboratory of Chemistry, California Institute of Technology.

Mr. J. O. Harrison, Jr., of the Eckert-Mauchly Division of Remington Rand, Incorporated, has accepted a position as Head of the Computation Laboratory, Operations Research Office, Johns Hopkins University, Chevy Chase, Maryland.

Mr. F. C. Hatfield of Blackburn College has been appointed to an instructorship at State Teachers College, Mankato, Minnesota.

Associate Professor I. I. Hirschman of Washington University has been at the Institute for Advanced Study during the academic year 1952–53.

Professor W. G. Hubert of the City College of the City of New York has retired with the title of Professor Emeritus.

Mr. C. A. Jacokes of Knox College has been appointed to an instructorship at North Park College.

Mr. C. E. Kerr, previously a graduate assistant at the University of Delaware, has been appointed to an instructorship at Lafayette College.

Dr. Nathan Keyfitz of the Dominion Bureau of Statistics, Ottawa, Canada, is spending the year 1953 in Djakarta, Indonesia, as a statistical consultant.

Mr. F. A. Kros, who was formerly at White Sands Proving Ground, Las Cruces, New Mexico, is engaged now as a data reduction engineer with Land-Air, Incorporated, Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. M. R. Laudante, previously a student at Polytechnic Institute of Brooklyn, is now a research analyst with North American Aviation, Los Angeles, California.

Mr. E. A. Lew of the Metropolitan Life Insurance Company has been promoted to the position of Associate Actuary and Statistician.

Mr. S. L. Lida, who has been at the Los Alamos Scientific Laboratory, has accepted a position as Applied Science Representative, International Business Machines Corporation, New York City.

Assistant Professor F. W. Light of Johns Hopkins University has a position now as Chief, Wound Assessment Branch, Biophysics Division, Chemical Corps Medical Laboratories, Army Chemical Center, Maryland.

Mr. L. I. Lowell, formerly a graduate student at Michigan State College, is employed as a junior engineer with Boeing Airplane Company, Seattle, Washington.

Mr. J. R. Macy, previously a student at the University of Chicago, has accepted a position as a member of the technical staff of Bell Telephone Laboratories, New York City.

Mr. G. W. Medlin, who was a part-time instructor at the University of North Carolina, has been appointed to an instructorship at Wake Forest College.

Mr. George Millman, who was employed previously as a mathematician with the Army Map Service, has transferred to the position of Analytical Statistician, Office of the Quartermaster General, Department of the Army, Washington, D. C.

Dr. Knox Millsaps has a position as a research physicist with the Wright-Patterson Air Force Base, Dayton, Ohio.

Mr. Dewey Moore, formerly with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has been appointed Physicist at the Naval Ordnance Laboratory, White Oak, Maryland.

Dr. J. T. Moore has been appointed to an assistant professorship at Georgia Institute of Technology.

Mr. B. J. Morse, who has been doing graduate work at Columbia University is now engaged as a mathematician with the Vitro Corporation of America, Verona, New Jersey.

Mr. P. M. Moskowitz, formerly a physicist at the New York Naval Shipyard, is with the California Research and Development Company, Livermore, California.

Dr. W. L. Murdock has accepted a position as Applied Science Representative with International Business Machines Corporation, New York City.

Mr. A. B. Neale, previously at the Naval Aircraft Factory, Philadelphia, Pennsylvania, has a position as a senior engineer with Marquardt Aircraft Company, Van Nuys, California.

Assistant Professor W. V. Neisius of Georgia Institute of Technology has accepted a position with Logistics Research Company, Redondo Beach, California.

Associate Professor Ruth E. O'Donnell of Duquesne University is employed now as Senior Scientist with the Atomic Power Division of Westinghouse Electric Corporation, Pittsburgh, Pennsylvania.

Mr. R. D. Oeder, formerly a research assistant at Los Alamos Scientific Laboratory, is a mathematician at the Radiation Laboratory, University of California at Livermore.

Mr. S. R. Orr, who has been a research physicist at the Mound Laboratories, Miamisburg, Ohio, has accepted a position as a staff member at the Los Alamos Scientific Laboratory.

Mr. W. D. Paxton, Jr., has a position as Aerodynamicist-Mathematician with North American Aviation, Los Angeles, California.

Mr. S. R. Peterson of the University of New Hampshire has been appointed to an instructorship at Union College.

Mr. R. M. Pinkerton, previously with the National Advisory Committee for Aeronautics, Langley Field, Virginia, has been appointed Professor of Mechanical Engineering at North Carolina State College.

Mr. C. A. Pursel of California State Polytechnic College has a position as a process engineer, General Electric Company, Richland, Washington.

Professor W. T. Reid has returned to Northwestern University after serving as a staff member at Sandia Corporation, Albuquerque, New Mexico.

Professor W. D. Rice of Olivet Nazarene College has been promoted to the position of Chairman of the Department of Physics and Mathematics.

Mr. F. P. J. Sansom, formerly a student at Baker University, has accepted a position at Wright Air Development Center, Dayton, Ohio.

Mr. Arthur Schach of the City College of the City of New York has been appointed Assistant Editor, United States Quarterly Book Review, Library of Congress, Washington, D. C.

Mr. Seymour Schuster has been appointed to an instructorship at Polytechnic Institute of Brooklyn effective September 1, 1953.

Dr. W. R. Seugling, who has been a research assistant at the University of California at Los Angeles, is engaged now as a structural engineer with Yoh Engineering, Incorporated, Los Angeles, California and Wichita, Kansas.

Reverend R. J. Swords, who has been a student at Rathfornham Castle, Dublin, Ireland, has been appointed to an assistant professorship at College of the Holy Cross.

Mr. A. V. Sylwester of California Concordia College is now at the Weather Observer School, Chanute Air Force Base, Illinois.

Mr. R. T. Tear, who has been in military service, has returned to his position as an instructor at Rensselaer Polytechnic Institute.

Assistant Professor B. T. Wade of Clemson Agricultural College has accepted a position as a mathematician at Aberdeen Proving Ground, Maryland.

Miss Frances E. Walsh of Ursuline College has been appointed to an instructorship at the College of Saint Teresa, Winona, Minnesota.

Miss Betty J. Whaley, previously a student at Tennessee Polytechnic Institute, is employed as an engineering aide with the General Electric Company, Schenectady, New York.

Associate Professor R. M. Whitmore of Southwestern University is now a staff member with the Sandia Corporation, Albuquerque, New Mexico.

Mr. R. F. Willis, who has been doing graduate work at Columbia University, has accepted a position as a mathematician with the Operations Research Group of Arthur D. Little, Incorporated, Cambridge, Massachusetts.

Mr. W. W. Youden, formerly a student at Brown University, is engaged as an electronic scientist in the Electronic Computers Laboratory, National Bureau of Standards, Washington, D. C.

Dr. D. F. Campbell of Chicago, Illinois, died on February 5, 1953. He was a charter member of the Association.

THE MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

NEW MEMBERS

Professor H. M. Gehman, Secretary-Treasurer, announces that the following 103 persons have been elected to membership by the Board of Governors on applications duly certified.

E. F. ASSMUS, Student, Oberlin College.

L. F. BABCOCK, Ed.M. (Harvard) Teacher, Woodmere Academy, Woodmere, N. Y.

J. H. BARRETT, Ph.D. (Texas) Asst. Professor, University of Delaware.

JOSEPH BASILE, C.E. (Brussels) General Manager, Ateliers de Constructions Basile, Brussels, Belgium.

M. ISOBEL BLYTH, Ph.D. (Michigan) Asst. Professor, Michigan State College.

R. C. BOLES, Ed.D. (Florida) Asso. Professor, Florence State Teachers College, Ala.

F. E. BRADLEY, Design Engineer, McDonnell Aircraft Corporation, St. Louis, Mo.

D. G. BRENNAN, Student, Massachusetts Institute of Technology.

E. H. BRYANT, M.S. (Mississippi) Head of Department, Northwest Mississippi Junior College, Senatobia, Miss.

JAMES WINFIELD BUTLER, M.A. (Kent State) Junior Development Engineer, Goodyear Aircraft, Akron, Ohio.

L. A. CANERS, M.A. (Manitoba) M.A. (Minnesota) Asst. Professor, St. Michael's College, Winooski, Vt.

MR. DALE CARPENTER, B.S.E.E. (Ohio Northern) Mathematics Supervisor, Board of Education, Los Angeles, Calif.

T. F. CARROLL, B.S. (Tulane) Assistant Engineer, Celotex Corporation, Marrero, La.

E. W. CHENEY, B.S. (Lehigh) Teaching Fellow, University of Kansas.

T. S. CHIHARA, M.S. (Purdue) Grad. Teaching Assistant, Purdue University.

R. A. CLARK, Ph.D. (M.I.T.) Asst. Professor, Case Institute of Technology.

PATRICIA A. CLINES, B.S. (Nazareth C.) Grad. Student, Catholic University.

J. W. COLBY, Student, University of Louisville.

G. R. COSTELLO, Aerophysics Project Engineer, Chance-Vought Aircraft, Dallas, Tex.

G. L. CRUMLEY, B.S. (Kansas S.T.C., Emporia) Grad. Student, Kansas State Teachers College, Emporia, Kan.

H. B. CURTIS, JR., M.A. (Arkansas) Instr., Texas Agricultural and Mechanical College.

BEVERLY J. DALE, B.A. (Buffalo) Engineering Computer, Bell Aircraft Corporation, Niagara Falls, N. Y.

C. H. DALTON, M.A. (Michigan) Asst. Professor, Southeast Missouri State College, Cape Girardeau, Mo.

R. B. DAVIS, Ph.D. (M.I.T.) Asst. Professor, University of New Hampshire.

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CALENDAR OF FUTURE MEETINGS

Thirty-fourth Summer Meeting, Queen's University and the Royal Military College, Kingston, Ontario, Canada, August 31–September 1, 1953.

Thirty-seventh Annual Meeting, Johns Hopkins University, Baltimore, Maryland, December 31, 1953.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Associate Secretary.

ALLEGHENY MOUNTAIN
ILLINOIS
INDIANA
IOWA
KANSAS
KENTUCKY
LOUISIANA-MISSISSIPPI, Southwestern Louisiana Institute, Lafayette, February 19–20, 1954.
MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA
METROPOLITAN NEW YORK
MICHIGAN
MINNESOTA, Bemidji State Teachers College, October 10, 1953.
MISSOURI
NEBRASKA
NORTHERN CALIFORNIA
OHIO
OKLAHOMA, Oklahoma City, October, 1953.
PACIFIC NORTHWEST, Montana State University, Missoula, June 19, 1953.
PHILADELPHIA, Drexel Institute of Technology, Philadelphia, November 28, 1953.
ROCKY MOUNTAIN
SOUTHEASTERN, University of South Carolina, Columbia, March 12–13, 1954.
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